Chapter 4.

Compound Interest

Compound interest.

The previous chapter commenced by asking how much interest would be earned after 1 year, 2 years or perhaps just 6 months, if \$500 were to be invested in an account paying interest at the rate of 10% per annum.

Considering a *simple interest* approach we saw that:

$$\begin{aligned} \text{Value after 2 years} &= \$500 + 2 \times 10\% \text{ of } \$500 \\ &= \$500 + 2 \times \$50 \\ &= \$600 \end{aligned}$$

With *compound interest* the interest is added to the account at the end of each *compounding period.* This interest will then itself earn interest in the second and subsequent compounding periods. This "interest on the interest" is the distinguishing feature of compound interest over simple interest.

$$\begin{aligned} \text{Value after 1 year} &= \$500 + 10\% \text{ of } \$500 \\ &= \$500 + \$50 \\ &= \$550 \quad (= \$500 \times 1{\cdot}1) \\ \text{Value after 2 years} &= \$550 + 10\% \text{ of } \$550 \\ &= \$550 + \$55 \\ &= \$605 \quad (= \$500 \times 1{\cdot}1 \times 1{\cdot}1) \\ &\qquad\qquad (\text{i.e. } \$500 \times 1{\cdot}1^2) \end{aligned}$$

The value at the end of each year is related to that of the previous year as follows:

$$\text{Value at end of one year} = \text{Value at start of that year} \times 1{\cdot}1$$

If we want the value after n years:

$$\text{Value after } n \text{ years} = \text{Initial value} \times 1{\cdot}1^n$$

If you managed to create the spreadsheet for simple interest mentioned in the previous chapter try to include the compound interest situation, as shown below.

	A	B	C	D	E	F	G	H
1	Amount invested ($)		**$500.00**					
2	Annual interest rate		**10.00%**					
3		Simple interest account.				Compound interest account.		
4	End of year	1	$550.00			1	$550.00	
5		2	$600.00			2	$605.00	
6		3	$650.00			3	$665.50	
7		4	$700.00			4	$732.05	
8		5	$750.00			5	$805.26	
9		6	$800.00			6	$885.78	
10		7	$850.00			7	$974.36	
11		8	$900.00			8	$1,071.79	

In the above situation compare the value after 8 years of simple interest,

i.e. $500 + 8 lots of $50 = $900,

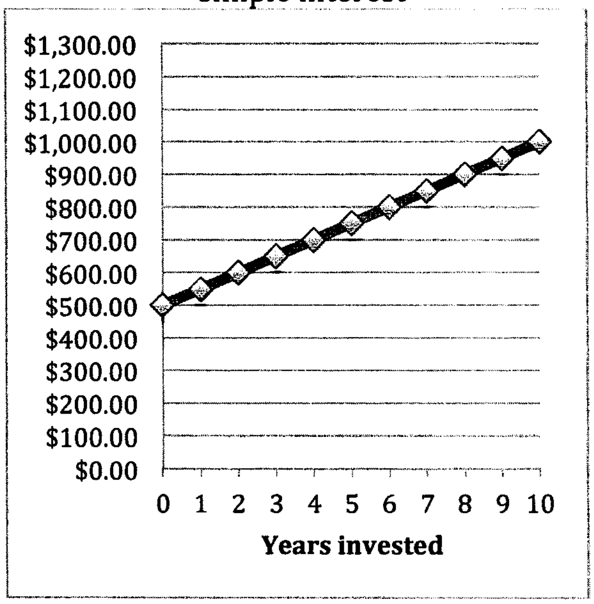
to the value after 8 years of compound interest,

i.e. $\$500 \times 1{\cdot}1^8 = \$1071{\cdot}79$.

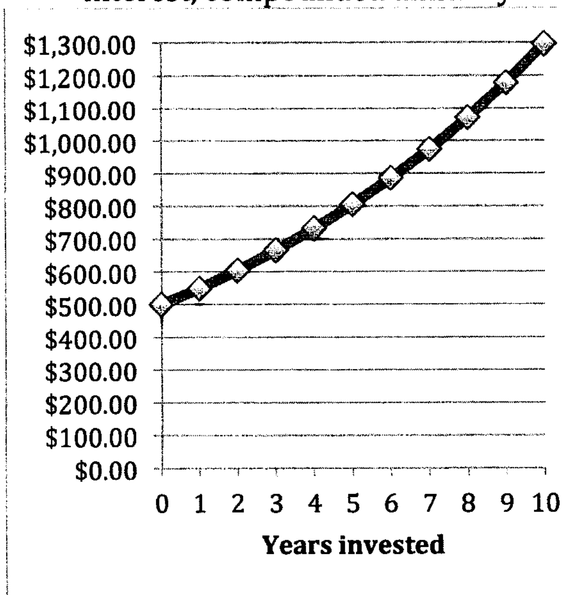
☛ Compare the same investment after 50 years of simple interest with the investment after 50 years of compound interest.

The data for each situation is shown graphed below. Notice that the simple interest situation shows a straight line or *linear* relationship whilst the compound interest situation shows what we call *exponential* growth.

$500 invested at 10% pa simple interest



$500 invested at 10% pa compound interest, compounded annually.

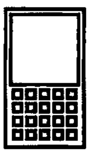


Example 1

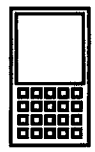
If \$2500 is invested at 8% per annum compound interest, with the interest compounded annually, how much is the investment worth after 4 years?

Value at end of 1st year	=	$\$2\,500 \times 1{\cdot}08$	=	$\$2\,700$
Value at end of 2nd year	=	$\$2\,700 \times 1{\cdot}08$	=	$\$2\,916$
Value at end of 3rd year	=	$\$2\,916 \times 1{\cdot}08$	=	$\$3\,149{\cdot}28$
Value at end of 4th year	=	$\$3\,149{\cdot}28 \times 1{\cdot}08$	=	$\$3\,401{\cdot}22$ (nearest cent)
Or, using indices:		$\$2500 \times 1{\cdot}08^4$	=	$\$3\,401{\cdot}22$ (nearest cent)

The value of the investment after 4 years is \$3 401·22, to the nearest cent.



Alternatively we could use the built in capability of some calculators, or some internet programs, to perform compound interest calculations.
Explore such capabilities.



Example 2

If \$12000 is invested at 6·2% per annum compound interest, with interest compounded annually, how much is the investment worth after 10 years?
Compare the final value with that of the same amount invested for the same time but with the 6·2% per annum calculated as simple interest.

Value at end of 10th year $= \$12\,000 \times 1{\cdot}062^{10}$
$= \$21\,899{\cdot}11$ (to the nearest cent)

The value of the investment after 10 years is \$21 899·11, to the nearest cent.

With simple interest, interest after 10 years $= \$12\,000 \times 0{\cdot}062 \times 10$
$= \$7\,440$

Hence with simple interest the final value $= \$12\,000 + \$7\,440$
$= \$19\,440$

After ten years the compound interest method returns \$2459·11 more than the simple interest method.

Other compounding periods.

In the compound interest situations encountered so far the interest has been compounded *annually*. Suppose that the 8% per annum of example 1 above were compounded every six months instead of every 12 months. The 8% per annum would now mean 4% per compounding period and in the 4 years there would be eight compounding periods. In this case:

Value of investment at end of 4th year $= \$2\,500 \times 1{\cdot}04^8 = \$3\,421{\cdot}42$.

The six monthly compounding has returned \$20·20 more interest after four years than the annual compounding did.

Example 3

Determine the interest paid on \$5000 invested at 8% per annum compound interest for 3 years, with the interest compounded

(a) annually,
(b) every six months,
(c) quarterly.

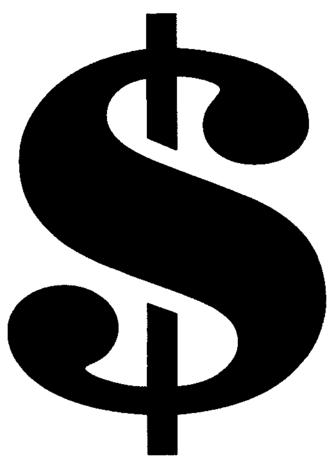
(a) Value of investment after 3 years $= \$5\,000 \times 1{\cdot}08^3$
$= \$6\,298{\cdot}56$
Thus the interest paid $= \$6\,298{\cdot}56 - \$5\,000$
$= \$1\,298{\cdot}56$

(b) If interest is compounded every 6 months the 8% per annum means 4% each compounding period.
In 3 years there will be 6 compounding periods.
Value of investment after 3 years $= \$5\,000 \times 1{\cdot}04^6$
$= \$6\,326{\cdot}60$ (to nearest cent)
Thus the interest paid $= \$6\,326{\cdot}60 - \$5\,000$
$= \$1\,326{\cdot}60$

(c) If interest is compounded quarterly (every 3 months) the 8% per annum means 2% each compounding period.
In 3 years there will be 12 compounding periods.
Value of investment after 3 years $= \$5\,000 \times 1{\cdot}02^{12}$
$= \$6\,341{\cdot}21$ (to nearest cent)
Thus the interest paid $= \$6\,341{\cdot}21 - \$5\,000$
$= \$1\,341{\cdot}21$

Borrowing money.

As mentioned in the chapter on simple interest, it is not always about investing money - sometimes we have to borrow money from a financial institution rather than lending money to it. We then have to pay interest to the institution for the money we have borrowed.



Suppose we borrow \$1000 for six months and the interest rate we are charged is 12% per annum.

Now 12% of \$1000 is \$120 so, using a simple interest approach, the total interest we would be charged for the six months is \$60.

If we use a monthly compound interest approach then the amount owed at the end of the 6 months is

$$\$1000 \times 1{\cdot}01^6.$$

Thus \$1061·52 is owed, i.e. interest of \$61·52.

Example 4
How much will be owed after 3 years on a loan of $4000 with compound interest charged at 8% per annum, compounded daily.

In 3 years there will be 3 × 365 (= 1095) compounding periods.

Value of investment after 3 years $= \$4000 \times \left(1 + \frac{0{\cdot}08}{365}\right)^{1095}$

$= 5084{\cdot}86$, to the nearest cent

Again explore the capability of your calculator, and of some internet programs, to perform calculations involving compound interest, especially when compounding occurs monthly or weekly.

Exercise 4A

1. If $5 000 is invested at 5% per annum compound interest with the interest compounded annually, how much is the investment worth after 4 years?

2. If $200 is invested at 8% per annum compound interest with the interest compounded annually, how much is the investment worth after 25 years?

3. How much is owed after three years if $250 is borrowed at 18% per annum compound interest with no repayments being made before the end of the three years and interest is compounded annually?

4. Determine the interest paid on $1000 invested at 4% per annum compound interest for 3 years, with the interest compounded
   (a) annually,
   (b) every six months,
   (c) quarterly.

5. Determine the interest paid on $5000 invested at 12% per annum compound interest for 3 years, with the interest compounded
   (a) annually,
   (b) every six months,
   (c) monthly.

6. How much is owed after two years if $2000 is borrowed at 6% per annum compound interest with interest compounded monthly and no repayments being made before the end of the two years?

7. Determine the value of an investment of $2000 after 2 years if the interest rate is 12% per yr calculated
   (a) as simple interest,
   (b) as compound interest compounded annually,
   (c) as compound interest compounded monthly,
   (d) as compound interest compounded daily.

8. A product called a "reverse mortgage" allows older people who own their own home to take out a loan to help their financial situation whilst on a pension. The amount that needs to be repaid increases with time as the interest payments are added but no repayments are made until the person taking out the loan dies. The home is then sold, the loan plus interest is repaid from the proceeds and any remaining funds are distributed according to the will of the deceased person.
   An elderly couple borrow $40 000 with interest compounded monthly at a rate of 9% per annum. How much will need to be repaid on this loan 15 years later?

9. Copy and complete the following table to compare various forms of interest for a loan of $10 000 at 8% per annum.
   (You may wish to use a spreadsheet on a computer or calculator.)

	$10000 borrowed at 8% per annum			
	Simple Interest	Compounded Annually	Compounded every 6 months	Compounded quarterly
Initial amount borrowed				
Amount owed after 1 year				
Amount owed after 2 years				
Amount owed after 3 years				
Amount owed after 4 years				
Amount owed after 10 years				
Amount owed after 20 years				

10. Copy and complete the following table to compare various forms of interest for an investment of $2 000 at 12% per annum.
    (You may wish to use a spreadsheet on a computer or calculator.)

	$2 000 invested at 12% per annum			
	Simple Interest	Compounded Annually	Compounded every 6 months	Compounded monthly
Initial balance				
Balance after 1 year				
Balance after 2 years				
Balance after 3 years				
Balance after 4 years				
Balance after 10 years				
Balance after 20 years				

Inflation and depreciation

Inflation was mentioned in chapter 2 as an example of the everyday use of percentages. It is again mentioned in this chapter (along with the concept of depreciation) because, whilst the concepts are not examples of compound interest, they each similarly involve the repeated multiplication by a number. For example repeated multiplication by 1·1 for a 10% inflation rate, or by 0·9 for a 10% depreciation rate.

Consider again the "reversible mortgage" situation explained in question 8 of the previous exercise:

> *A product called a "reverse mortgage" allows older people who own their own home to take out a loan to help their financial situation whilst on a pension. The amount that needs to be repaid increases with time as the interest payments are added but no repayments are made until the person taking out the loan dies. The home is then sold, the loan plus interest is repaid from the proceeds and any remaining funds are distributed according to the will of the deceased person.*

The elderly couple may be alarmed that their initial loan of $40000 "blows out" to more than $150000 ($\$40000 \times 1{\cdot}0075^{180}$) after 15 years because they are allowing the interest to accumulate rather than reducing it by making repayments. However in this fifteen years the house that will be sold to pay off the loan will have **appreciated** (increased) in value due to **inflation** (the general rise in prices with time). Suppose their house was worth $310 000 at the start of the fifteen years and that the annual inflation rate is 3·4%.

Value of the house after 15 years = $\$310\,000 \times 1{\cdot}034^{15}$
= $512 000 to the nearest $1000.

Hence whilst the loan has increased by approximately $110 000 in the fifteen years the asset that will be used to pay off the loan has increased by approximately $202 000 in the same time due to inflation.

✍ Consider this reverse mortgage situation with the $40000 borrowed at 12% per annum compounded monthly for 25 years, and house values rising at just 3% per annum.

Example 5

A particular item has a current value of $8 600, this value rising in line with inflation. If we assume a constant annual inflation rate of 4·2% what will the value of this item be

(a) 2 years from now (b) 25 years from now?

With an annual inflation rate of 4·2% we have annual multiplication by 1·042.

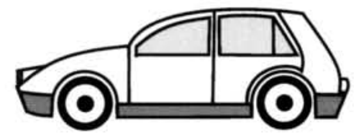
(a) Value of the item 2 years from now = $\$8\,600 \times 1{\cdot}042^{2}$
= $9 337·57 to the nearest cent.

Assuming a constant annual inflation rate of 4·2% the item will have a value of approximately $9 300 two years from now.

(b) Value of the item 25 years from now = $\$8\,600 \times 1{\cdot}042^{25}$
= $24 054·23 to the nearest cent.

Assuming a constant annual inflation rate of 4·2% the item will have a value of approximately $24 000 twenty five years from now.

Not everything increases in value. Some items fall in value as they get older. They are said to **depreciate**. A car for example will fall in value as it gets older (until it becomes so old that it becomes rare and collectible in which case its value could start to rise).



Example 6

A particular car has a new value of \$28 000. If we assume a constant annual depreciation rate of 7% what will the value of this car when it is (a) 2 years old, (b) 8 years old.

With a constant annual depreciation rate of 7% we have annual multiplication by 0·93.

(a) Value of the car when 2 years old = $\$28\,000 \times 0{\cdot}93^2$

= \$24 217·20 to the nearest cent.

Assuming a constant annual depreciation rate of 7% the car will have a value of approximately \$24 200 when it is two years old.

(b) Value of the car when 8 years old = $\$28\,000 \times 0{\cdot}93^8$

= \$15 668·29 to the nearest cent.

Assuming a constant annual depreciation rate of 7% the car will have a value of approximately \$15 700 when it is eight years old.

Exercise 4B

1. A particular car has a new value of \$32 000. If we assume a constant annual depreciation rate of 12% what will the value of the car be when it is
   (a) 1 year old, (b) 5 years old.

2. A particular house has a current value of \$350 000. If we assume a constant annual inflation rate of 4·8% what will the value of this house be
   (a) 2 years from now, (b) 20 years from now, (c) 50 years from now?

3. Assuming a constant annual inflation rate of 4% what will be the cost of a particular type of chocolate bar in 20 years time if it costs \$2·20 now?
   Suppose instead that the annual inflation rate for the period was 8% rather than 4%. What would the chocolate bar cost in 20 years time now?

4. A car has a current value of \$32000. If we assume a constant depreciation rate of 7·2% per year what will the value of this car be
   (a) 3 years from now, (b) 5 years from now, (c) 10 years from now?

5. The increased availability of a particular commodity causes its price per kg to depreciate by 5·2% each year for a five year period. At the start of this period the commodity cost \$135 per kg. What was the cost per kg of this commodity at the end of the five year period?

Miscellaneous Exercise Four.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary section at the beginning of the book.

1. (a) Find 20% of \$40 (b) Find 40% of \$20
   (c) Find 30% of \$120 (d) Find 5·8% of \$120
   (e) Increase \$750 by 16% (f) Decrease \$430 by 18%.

2. The number of years, T, required for an investment of \$P, earning simple interest at a rate of R% per annum, to earn \$I interest can be found using the formula:

$$T = \frac{100 \times I}{PR}$$

   Find T if (a) P = 300, R = 5, I = 90 (b) P = 540, R = 7·5, I = 324
   (c) P = 75·80, R = 5, I = 37·90 (d) P = 17500, R = 4, P + I = 19950

3. If inflation is running at a steady 4·4% per annum, and considering inflation to be the only reason for prices to rise, what will be the cost of an item in ten years if its cost now is \$240?

4. Anje wishes to invest \$8000 for 3 years. She considers three schemes:
   A: Simple interest at 9·00% per annum.
   B: Compound interest at 8·15% per annum compounded quarterly.
   C: Compound interest at 8·08% per annum compounded monthly.
   Which scheme should she choose to maximize the value of the account at the end of the three year period and what would that maximum value be?

5. An elderly couple who own their own home decide to borrow \$60 000.
   The loan, plus interest, is to be paid off from the proceeds of the sale of the house upon the death of the last surviving member of the couple (or earlier if the couple decide to sell the house earlier).
   Interest is to be compounded annually and the interest rate is fixed at 8·5% per annum.
   The table below shows the interest to be added and the amount owing at the end of a number of years.

Year	Interest for the year	Loan amount
1	\$5100	\$65 100
2	\$5533·50	\$70 633·50
3		
4		
10		
25		

   Copy and complete the table.

6. Tax changes announced.
Suppose that the government of a country for which the income tax rates were as given in the table below left announces that as from the following tax year the rates will change to those shown in the table below right:

Current tax year.	
Taxable income	Tax rate
$0 to $19 400	0%
$19 401 to $37 000	19%
$37 001 to $80 000	33%
$80 001 to $180 000	37%
$180 001+	45%

Next tax year.	
Taxable income	Tax rate
$0 to $24 000	0%
$24 001 to $45 000	15%
$45 001 to $90 000	30%
$90 001 to $180 000	38%
$180 001+	46%

Suppose that you work for a newspaper and, for a proposed article about the tax changes, you are asked to produce:

✎ A table like that shown below for the current tax year but your table should be for the "next tax year" rates.

Taxable income	Income tax to pay
$0 to $19 400	Nil
$19 401 to $37 000	19% of the taxable income over $19400
$37 001 to $80 000	$3344 + 33% of the taxable income over $37000
$80 001 to $180 000	$17534 + 37% of the taxable income over $80000
$180 001+	$54534 + 45% of the taxable income over $180000

AND

✎ A table showing what people on various taxable incomes will save, both over a full year and per week, when the new rates are applied. I.e. a table like that shown below:

What the tax changes will mean to you.		
Taxable income.	Savings per year.	Savings per week.
$5 000		
$15 000		
$25 000		
$35 000		
$50 000		
$60 000		
$75 000		
$100 000		
$125 000		
$150 000		
$200 000		

Produce completed tables as required.

Challenge: For what taxable incomes would the new rates mean you would be paying more total income tax than under the old rates.