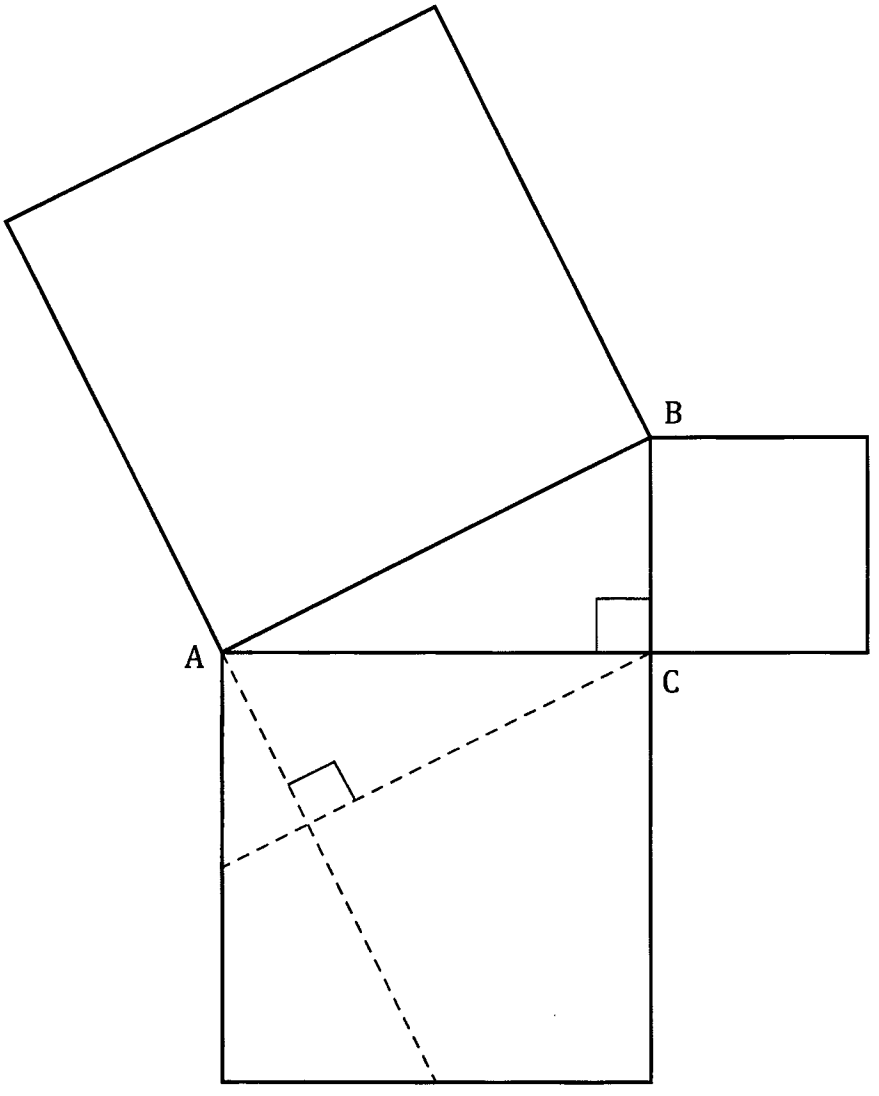
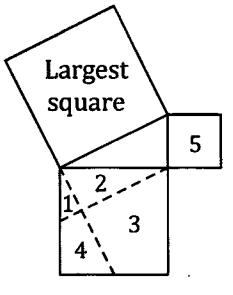
Chapter 7.

The theorem of Pythagoras.

Accurately draw a right triangle on a piece of A4 or A3 paper making the right triangle as large as you can whilst still leaving enough room to accurately draw a square on each side of the triangle, as shown below for triangle ABC.



Now draw lines across the middle sized square, as shown by the broken lines in the diagram above. Cut out this square and accurately cut it up into the four pieces, 1, 2, 3 and 4 as indicated by the diagram on the right. Also cut out the small square, shape 5 in the diagram on the right. Try to fit shapes 1 to 5 together on the largest square to exactly fill it.



Pythagoras' theorem.

Pythagoras was a Greek philosopher who lived approximately two and a half thousand years ago. One of the things he is most famous for is the result simply known as the theorem of Pythagoras (or Pythagoras' theorem or the Pythagorean theorem).

Note: ☞ A *philosopher* is a person who seeks to understand, and gain knowledge about, the causes and underlying principles of things.

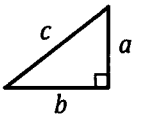
☞ A *theorem* is a statement, the truth of which can be established using existing truths.

If you successfully managed the activity on the previous page you demonstrated the truth of Pythagoras' theorem:

The square of the length of the longest side of a right angled triangle is equal to the sum of the squares of the lengths of the other two sides.

Note: ☞ The longest side of a right triangle is the one opposite the right angle and is called the *hypotenuse.*

Thus, for the triangle shown on the right, $c^2 = a^2 + b^2$

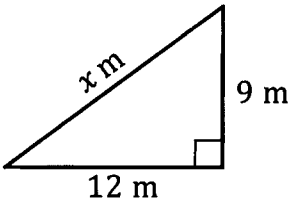


Using the Pythagorean theorem to calculate the sides of right triangles.

The theorem of Pythagoras allows us to determine the length of one side of a right triangle, knowing the lengths of the other two sides, as the following examples demonstrate.

Example 1

The right triangle sketched on the right has a hypotenuse of length x m. Find the value of x.



Using Pythagoras' theorem: $x^2 = 12^2 + 9^2$

$= 144 + 81$

$= 225$

Solving gives $x = 15$ (Negative solution not applicable.)

Thus $x = 15$

Note: • Asked to solve the equation $x^2 = 225$ there are <u>two</u> answers:

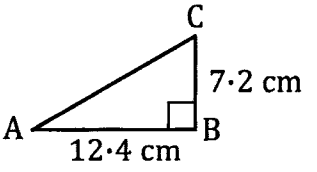
$x = 15$ or -15 ($15^2 = 225$ and $(-15)^2 = 225$)

However, in this context x m is the length of one side of a triangle so a negative value for x is not possible. Hence the inclusion of the statement in the above solution, *Negative solution not applicable.*

- Notice also that the question gives the length of the hypotenuse as x m. Hence x is a number, in this case a number of metres. In our answer we say $x = 15$. It would be incorrect to say $x = 15$ m as the length of the hypotenuse would then be 15 metre metres!

Example 2

Find the length of side AC in the right triangle shown on the right, giving your answer in centimetres, correct to the nearest millimetre.



Using Pythagoras' theorem: $AC^2 = AB^2 + BC^2$

Thus if AC is of length x cm: $x^2 = 12{\cdot}4^2 + 7{\cdot}2^2$

$= 205{\cdot}6$

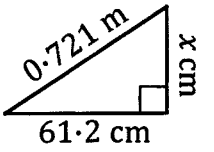
Solving gives $x = 14{\cdot}34$ correct to 2 decimal places

(Negative solution not applicable.)

Side AC is of length 14·3 cm, to the nearest millimetre.

Example 3

Find the value of x in the diagram shown on the right, giving your answer correct to 1 decimal place.



Using Pythagoras' theorem: $61{\cdot}2 + x^2 = 72{\cdot}1^2$

Thus $x^2 = 72{\cdot}1^2 - 61{\cdot}2^2$

$= 1452{\cdot}97$

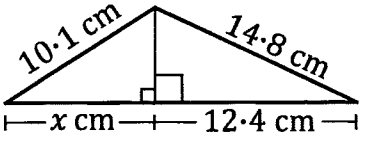
Solving gives $x \approx 38{\cdot}12$ correct to 2 decimal places

(Negative solution not applicable.)

Thus $x = 38{\cdot}1$ correct to 1 decimal place.

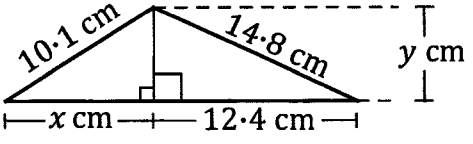
Example 4

Find the value of x in the diagram shown on the right, giving your answer correct to one decimal place.



Let the height of the triangle be y cm (see diagram).

Applying Pythagoras to the further right of the two triangles: $14{\cdot}8^2 = y^2 + 12{\cdot}4^2$



Hence $y^2 = 14{\cdot}8^2 - 12{\cdot}4^2$

$= 65{\cdot}28$

(i.e. $y = 8{\cdot}1$ correct to 1 decimal place.)

Now, from the triangle on the left of the two,

$10{\cdot}1^2 = y^2 + x^2$

$= 65{\cdot}28 + x^2$

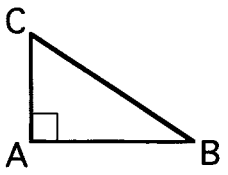
Solving gives $x \approx 6{\cdot}061$

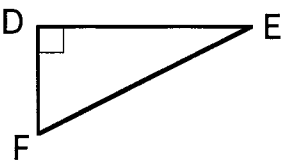
$= 6{\cdot}1$ correct to 1 decimal place.

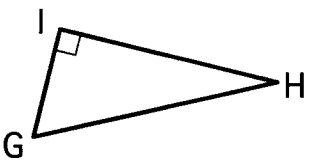
Notice that the calculation for x used the accurate value for y, i.e. $\sqrt{65{\cdot}28}$, rather than the 1 d.p. rounded value of 8·1 and thus avoided "rounding errors".

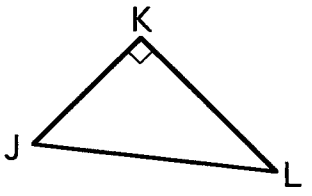
Exercise 7A.

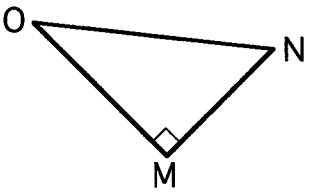
1. State the hypotenuse in each of the following triangles.

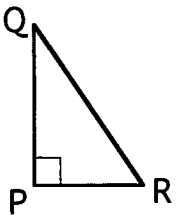
(a) 

(b) 

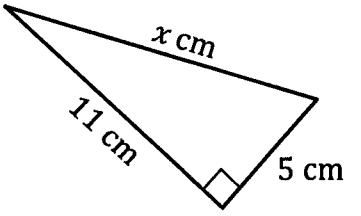
(c) 

(d) 

(e) 

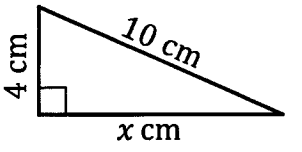
(f) 

2. For the right triangle shown on the right which of the following statements are true? (There may be more than one correct statement.)



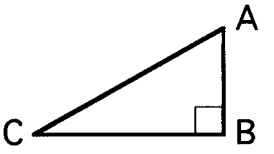
Statement I	Statement II	Statement III
$11^2 + x^2 = 5^2$	$x^2 + 11^2 = 5^2$	$11^2 + 5^2 = x^2$
Statement IV	**Statement V**	**Statement VI**
$5^2 + x^2 = 11^2$	$x^2 + 5^2 = 11^2$	$5^2 + 11^2 = x^2$

3. For the right triangle shown on the right which of the following statements are true? (There may be more than one correct statement.)



Statement I	Statement II	Statement III
$10^2 + 4^2 = x^2$	$10^2 + x^2 = 4^2$	$x^2 + 4^2 = 10^2$
Statement IV	**Statement V**	**Statement VI**
$x^2 = 4^2 - 10^2$	$x^2 = 10^2 - 4^2$	$4^2 = 10^2 - x^2$

4. For the right triangle shown on the right which of the following statements are true? (There may be more than one correct statement.)



Statement I	Statement II	Statement III
$AC^2 = AB^2 + BC^2$	$AB^2 = AC^2 + CB^2$	$BC^2 = BA^2 + AC^2$
Statement IV	**Statement V**	**Statement VI**
$AC^2 = BC^2 - AB^2$	$BC^2 = AC^2 - AB^2$	$AB^2 = AC^2 - BC^2$

Find the value of x in each of the following, rounding your answers to 1 decimal place if rounding is necessary.

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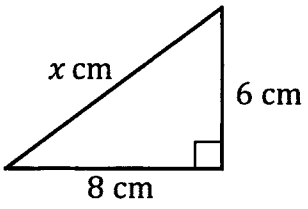

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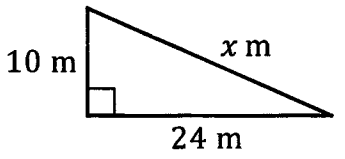

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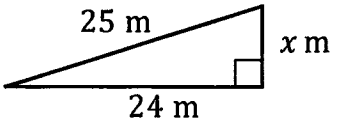

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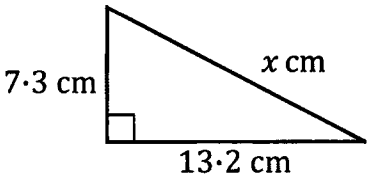

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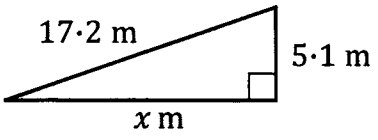

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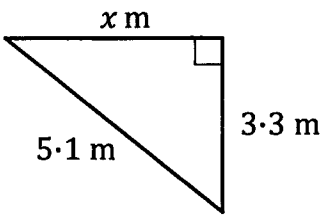

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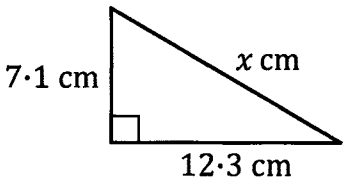

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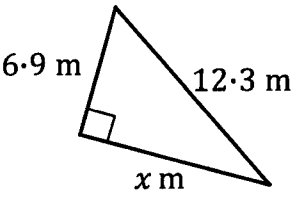

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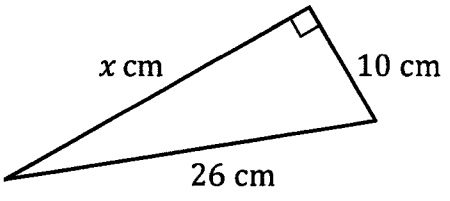

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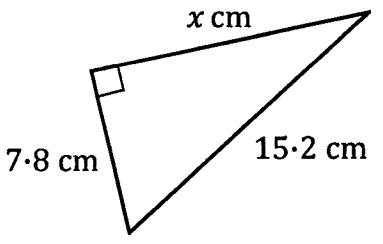

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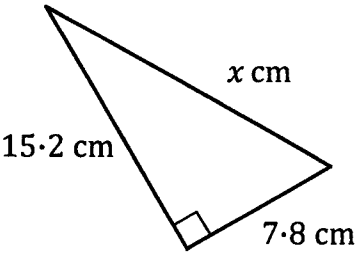

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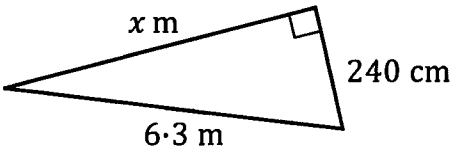

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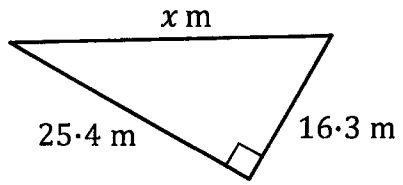

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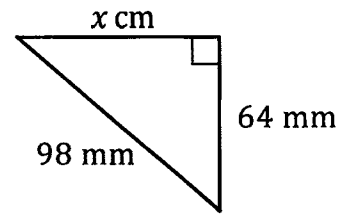

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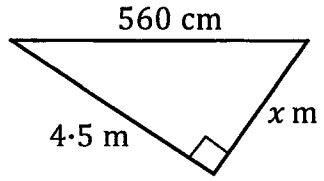

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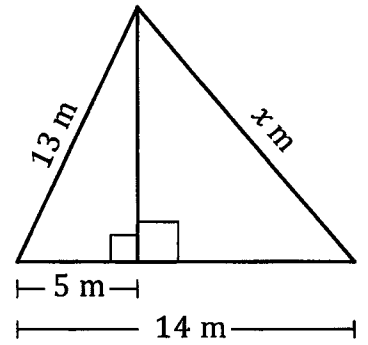

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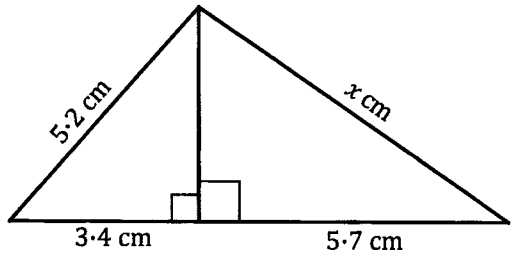

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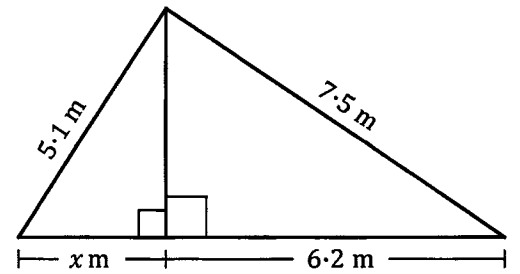

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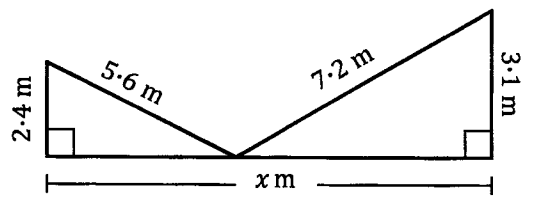

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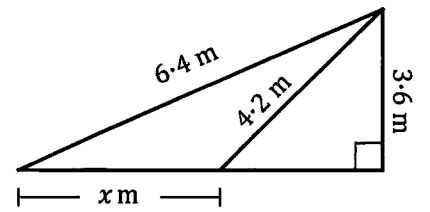
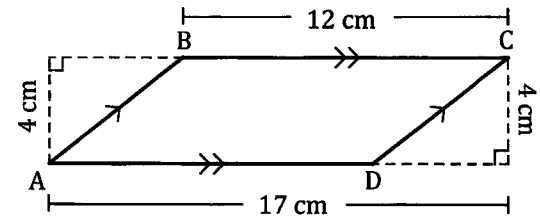


25. Triangle ABC is right angled at B and has AB = 26 mm and BC = 53 mm.
Find the length of AC, to the nearest millimetre.

26. Triangle PQR is right angled at R and has PQ = 17·3 cm and PR = 6·4 cm.
Find the length of RQ, to the nearest millimetre.

27. Triangle XYZ is a right triangle and has XY = 12·4 cm and YZ = 72 mm.
Find the possible lengths of XZ, to the nearest millimetre.

28. The diagram on the right shows the parallelogram ABCD with all lengths as indicated.
By how much does the longer diagonal of the parallelogram ABCD exceed the shorter diagonal? (Answer to the nearest millimetre.)

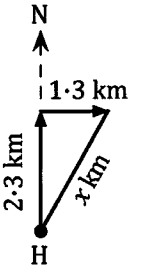


Applications.

The questions of the previous exercise all involved *abstract* right triangles - the triangles were not part of any practical situation. In the next examples, and the exercise that follows, the questions involve some real context. If such questions do not give a diagram of the situation you should draw one of your own to help you comprehend the question. Include given information on the diagram, introduce any letters you use and always make sure that your final statement does answer the question.

Example 5

A boat leaves a harbour, H, and travels 2·3 km due North and then 1·3 km due East. How far is the boat then from the harbour?



First draw a diagram and include on it any information that seems relevant.
Let the boat end up x km from the harbour.

By Pythagoras' theorem: $x^2 = 2{\cdot}3^2 + 1{\cdot}3^2$

$\therefore \quad x = \sqrt{2{\cdot}3^2 + 1{\cdot}3^2}$ (–ve sol^n not applicable)

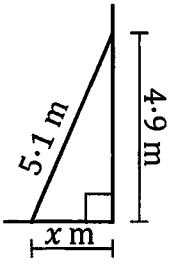
$\approx 2{\cdot}64$

The boat is then 2·6 km from the harbour (correct to 1 decimal place).

Notice that the final answer was not simply $x = 2{\cdot}64$. The letter x was not part of the initial problem, we introduced it to help obtain a solution. Interpreting the value of x back into the given situation, the final statement is given in terms of this situation and answers the question that was asked.

Example 6

A ladder has its base on level ground and its top resting against a vertical wall. If the ladder is 5·1 metres in length and reaches 4·9 metres up the wall how far is the foot of the ladder from the base of the wall?



First draw a diagram and include on it any information that seems relevant.
Suppose that the foot of the ladder is x m from the base of the wall.

By Pythagoras' theorem: $x^2 + 4{\cdot}9^2 = 5{\cdot}1^2$

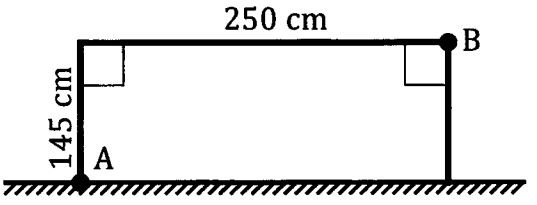
$\therefore \quad x^2 = 5{\cdot}1^2 - 4{\cdot}9^2$

Solving and rejecting the –ve sol^n gives $x \approx 1{\cdot}41$

The foot of the ladder is approximately 1·4 metres from the base of the wall.

Exercise 7B

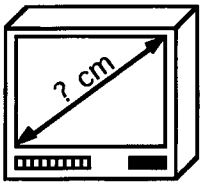
1. A show jumping fence is to be strengthened using a piece of timber from A to B, see diagram.
What length of timber is needed (to the nearest centimetre)?



2. A boat leaves a harbour and travels 5·63 km due West and then 1·32 km due North. How far is the boat then from the harbour? (Give your answer in km, rounded to one decimal place.)

3. A boat leaves a harbour and travels due East and then 2·12 km due South. The boat is then 3·54 km from the harbour. How far did the boat travel due east? (Give your answer in km, rounded to one decimal place.)

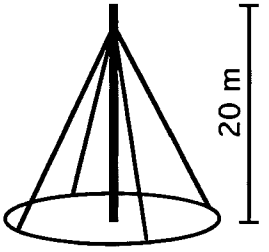
4. The size of a television screen is given as the size of the diagonal of the rectangular screen. What is the size of a rectangular screen with dimensions 56 cm by 42 cm?



5. A ladder has its base on level ground and its top resting against a vertical wall. If the ladder is 8·5 metres in length and reaches 7·8 metres up the wall how far, to the nearest 10 cm, is the foot of the ladder from the base of the wall?

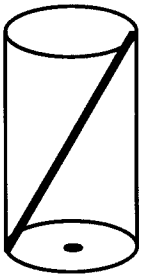
6. With its base on horizontal ground and 1·85 metres from a vertical wall a ladder just reaches the top of the wall. If the ladder is 5·25 metres in length find the height of the wall (to the nearest 10 centimetres).

7. A twenty metre mast is to be supported by four wires each with one end attached to the mast at a point two metres from the top and the other end attached to a point level with the base of the mast and five metres from it.
Rounding up to the next metre, what will be the length of each wire?

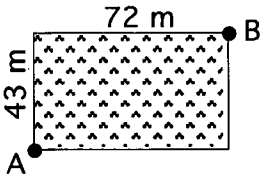


8. Ayetown lies 24 km due North of Ceetown.
Beetown lies 17 km due East of Ayetown.
A straight road links Ceetown to Ayetown and a straight road links Ayetown to Beetown.
A new straight road is planned to link Ceetown directly to Beetown.
Find how much this new road will reduce the journey from Ceetown to Beetown giving your answer in kilometres, correct to one decimal place.

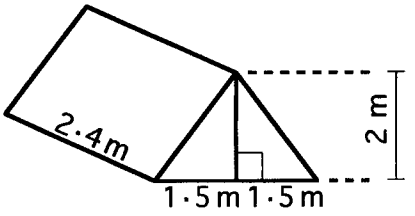
9. A rod of length 1 metre just fits inside a cylindrical container of base radius 25 cm.
Determine the height of the container, to the nearest cm?



10. Rather than use the footpath to travel around a rectangular park from point A to point B (see diagram) many people cut across the grass to travel in a straight line from A to B. How much shorter does this make the journey from A to B (to the nearest metre).



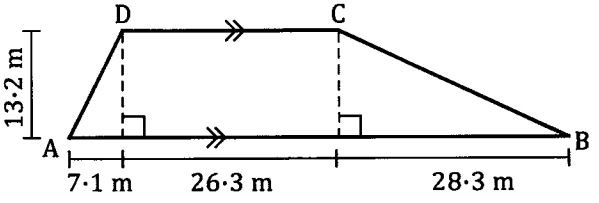
11. The diagram on the right shows a tent with all dimensions as indicated.
What is the area of each of the two rectangular sides of the tent?



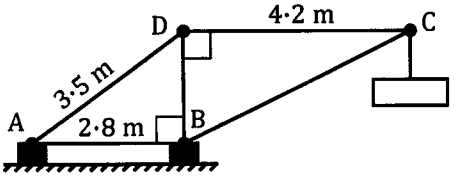
12. A company logo consists of a white square in a black circle with each corner of the square touching the edge of the circle.
If the circle has a diameter of 12 cm what is the length of each side of the square, in millimetres, to the nearest millimetre.

13. A company is asked to quote a price for making 100 steel frames each in the shape of a right triangle with one side of length 1·2 metres and the hypotenuse of length 1·8 metres.
In order to make the quote the company needs to know, amongst other things, the length of steel required to make the 100 frames.
Calculate this total length rounding your answer up to the next whole metre.

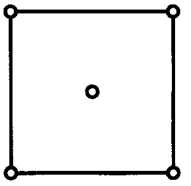
14. Fencing is to be placed around a play area that is in the shape of the trapezium ABCD shown in the diagram on the right.
With lengths as shown in the diagram what is the perimeter of the trapezium? (Give your answer in metres correct to one decimal place.)

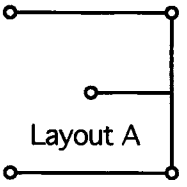


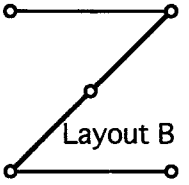
15. A crane has the basic structure shown on the right with all lengths as indicated.
Find the length of BC in metres, correct to one decimal place.



16. A watering system is to cover a square area of side length 8 metres with five sprinklers, one at each corner of the square and one at the centre. Which of the following systems of pipes uses the smaller total length of piping and how much smaller, to the nearest centimetre.



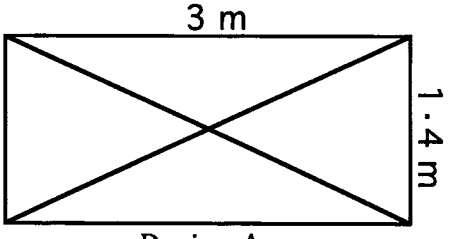




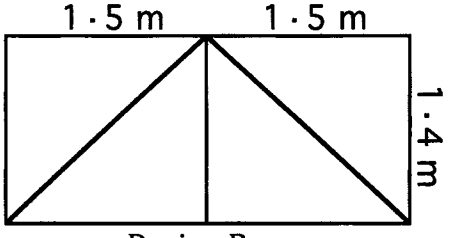
17. An area of land is in the shape of a right triangle with its hypotenuse of length 243·32 metres and one side of length 72·14 metres. Determine the area of this triangle, giving your answer to the nearest ten square metres.

[Remember the area of a triangle is $\frac{\text{Base} \times \text{Perpendicular height}}{2}$]

18. A rectangular frame measuring 3 metres by 1·4 metres is to be made of steel rods. Which of the two designs shown below requires the greater length of steel and by how much (to the nearest centimetre).

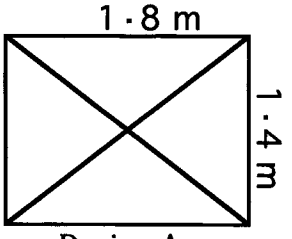


Design A

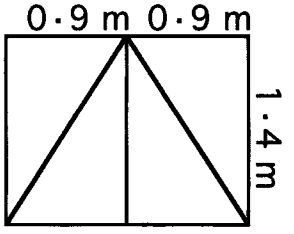


Design B

Suppose the frame was 1·8 metres by 1·4 metres instead.



Design A



Design B

Now which design requires the greater length of steel and by how much? (Again give your answer to the nearest centimetre).

3 ✷ 4 ✷ 5	5 ✷ 12 ✷ 13
Investigate Pythagorean triples.	
7 ✷ 24 ✷ 25	8 ✷ 15 ✷ 17

Miscellaneous Exercise Seven.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary section at the beginning of the book.

1. Increase \$1000 by 15% and then decrease your answer by 15%.

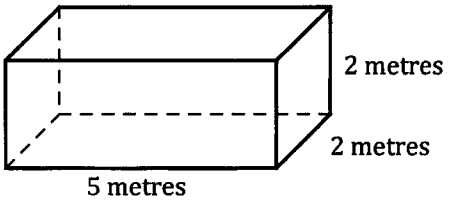
2. (a) Premultiply $\begin{bmatrix} 1 & -2 \end{bmatrix}$ by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. (b) Postmultiply $\begin{bmatrix} 1 & -2 \end{bmatrix}$ by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3. (a) Increase \$409·50 by 30%.
   (b) When an initial amount is increased by 30% it becomes \$409·50. What was the initial amount?
   (c) Decrease \$409·50 by 30%.
   (d) When an initial amount is decreased by 30% it becomes \$409·50. What was the initial amount?
   (e) Find 30% of \$409·50.

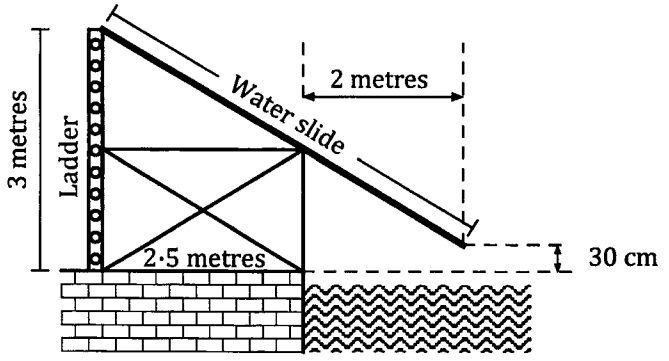
4. The matrices A, B and C shown below can be multiplied together to form a single matrix if A, B and C are placed in an appropriate order. What is the order and what is the single matrix this order produces.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 3 & 1 & 4 & 0 \end{bmatrix}.$$

5. What is the longest pole that could fit into the rectangular container shown on the right? (Give your answer to the nearest cm.)



6. The diagram below shows a vertical ladder leading to a water slide.



With all measurements as shown in the diagram determine the length of the water slide giving your answer to the nearest centimetre.

7. The route matrix on the right is for a road system between five points A, B, C, D and E. Explain how you can tell that the only "one way roads" are from A to C and from E to D.

		To A	B	C	D	E
	A	0	1	1	2	0
	B	1	0	1	3	0
From	C	0	1	0	1	1
	D	2	3	1	0	0
	E	0	0	1	1	0

8. One common form of matrix display is the *two way classification table*, an example of which is shown on the right. This table classifies the 1237 members of a health club according to two categories, gender, i.e. male or female, and age, "under 30" or "30 or over". To check that you understand the table, check that you agree with the following statements:

	Under 30	30 or Over	Totals
Male	329	276	605
Female	414	218	632
Totals	743	494	1237

743 of the health club members were under 30.

There were 27 more female members than male members.

(a) How many male club members are aged 30 or over?

(b) To the nearest percent what percentage of the female members were under 30?

(c) To the nearest percent what percentage of the club members were males under 30?

Note: These two way tables are also known as Carroll diagrams, named after the mathematician Charles Dodgson. So why are they named Carroll diagrams and not Dodgson diagrams? Do some research to answer this question.

9. Let us suppose that you are informed by email that a distant relative of yours passed away many years ago and it has just been realized that during her life she invested $100 into an account and now, 150 years later, you are the beneficiary of the account plus interest accrued. Now whilst most people would assume such information would simply be "an internet scam" let us suppose that in this case this most unlikely circumstance was in fact true. How much would this account plus interest now be worth if the $100 was invested in an account paying

(a) Simple interest of 15% per annum.

(b) Compound interest of 8% per annum compounded annually.

(c) Compound interest of 8% per annum compounded quarterly.

(d) Compound interest of 9% per annum compounded annually.