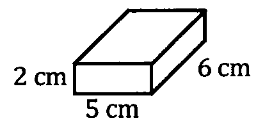
Chapter 10.
Similarity.

For the rectangular prism shown on the right:



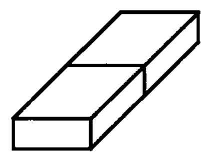
☞ The dimensions are 2 cm by 5 cm by 6 cm

☞ The surface area $= 2 \times 5\text{ cm} \times 6\text{ cm} + 2 \times 2\text{ cm} \times 5\text{ cm} + 2 \times 2\text{cm} \times 6\text{ cm}$
$= 60\text{ cm}^2 + 20\text{ cm}^2 + 24\text{ cm}^2$
$= 104\text{ cm}^2$

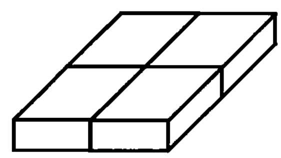
☞ The volume $= 2\text{ cm} \times 5\text{ cm} \times 6\text{ cm}$
$= 60\text{ cm}^3$

Now suppose we make a new prism that is

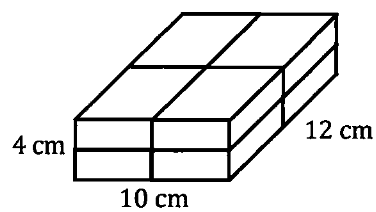
twice as long as the original:



as well as being twice as wide as the original:



as well as being twice as high as the original:



☞ The dimensions are now 4 cm by 10 cm by 12 cm

☞ The surface area $= 2 \times 10\text{ cm} \times 12\text{ cm} + 2 \times 4\text{ cm} \times 10\text{ cm} + 2 \times 4\text{cm} \times 12\text{ cm}$
$= 416\text{ cm}^2$

☞ The volume $= 4\text{ cm} \times 10\text{ cm} \times 12\text{ cm}$
$= 480\text{ cm}^3$

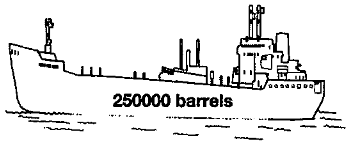
Thus while	original lengths	:	final lengths	= 1 : 2,		
the ratio	original surface area	:	final surface area	= 1 : 4	i.e.	$1 : 2^2$
and	original volume	:	final volume	= 1 : 8	i.e.	$1 : 2^3$

If one three dimensional object is an enlargement of another such that the ratio

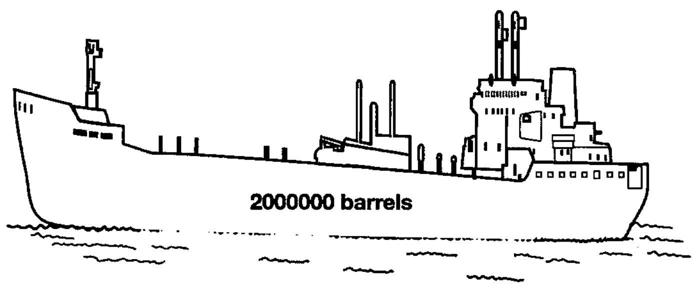
	lengths in original object :	corresponding lengths in enlargement	= $1 : k$
then	surface area of original :	surface area of enlargement	= $1 : k^2$
and	volume of original :	volume of enlargement	= $1 : k^3$

The quantity k is called the scale factor of the enlargement.

For example, suppose we have an oil tanker that is capable of carrying 250000 barrels of oil.

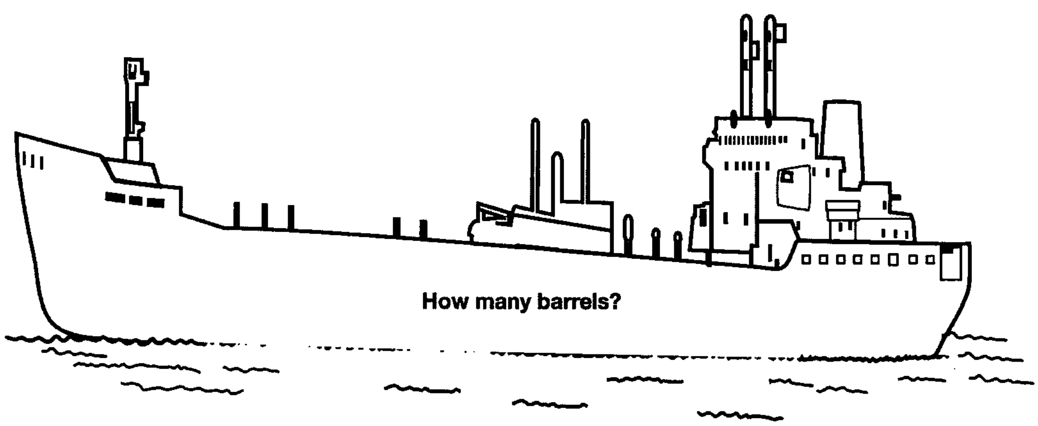


If we made a bigger version of this tanker with all lengths 2 times as long as they are in the original (i.e. the scale factor is 2, and the new tanker will be twice as long, twice as wide and twice as high as the original) the volume of the bigger tanker will be 8 (= 2^3) times the volume of the original.



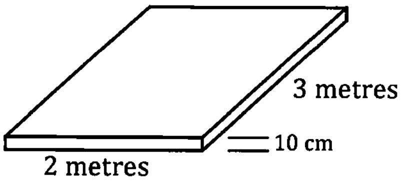
And the area of sheet metal required to make this bigger tanker will be 4 (= 2^2) times the area of sheet metal required to make the original.

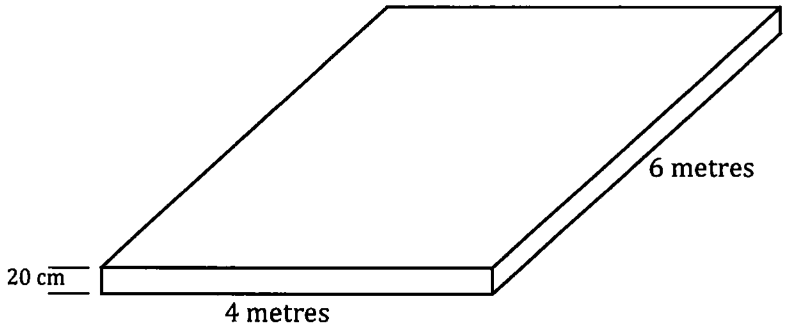
If we could make a tanker with all lengths 3 times the original lengths (scale factor of 3) then the volume of this bigger tanker will be 27 (= 3^3) times the volume of the original!



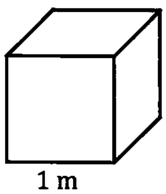
Exercise 10A.

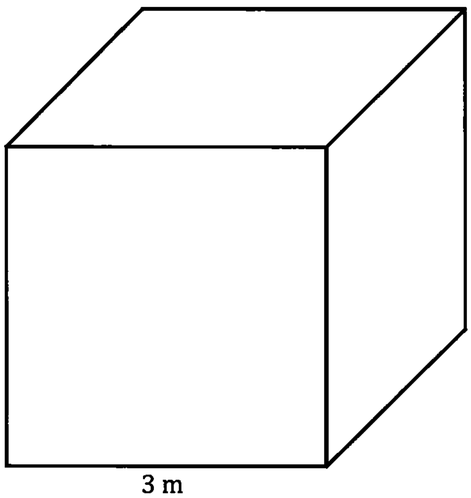
1. The cost of the concrete required to make the slab shown below left is \$120.
What does this suggest the cost of the concrete required to make the slab shown below right would be?



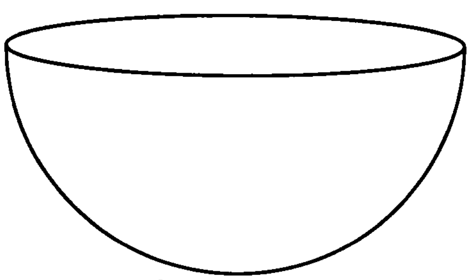


2. The cost of a particular type of paint required to paint the outside of the solid cube shown below left is \$30. What does this suggest would be the cost of the same type of paint used to paint the solid cube shown below right?

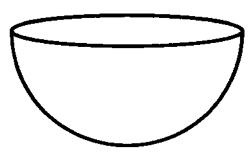




3. It required 2 hours to fill the hemisphere shown below left with water.
What does this suggest for the time the same supply would take to fill the hemisphere below right?

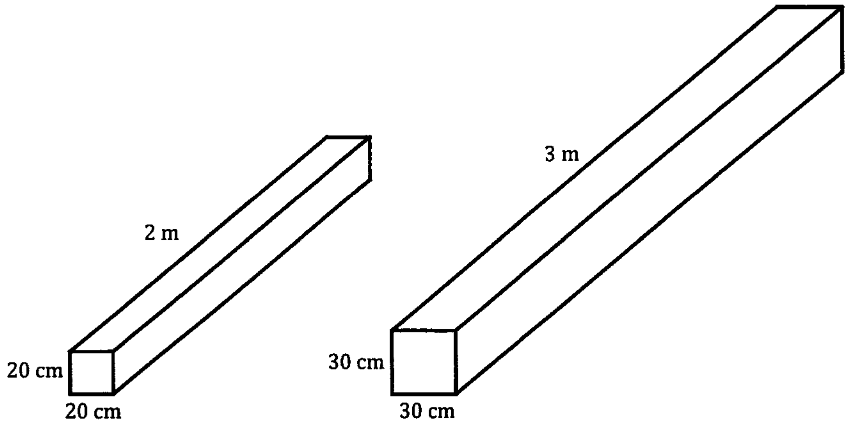


Hemisphere, radius 2 metres



Hemisphere, radius 1 metres

4. The cost of electro-plating all surfaces of the solid rectangular girder shown below left was $120.
What does this suggest would be the price of electro-plating the girder shown below right with the same material?



5. The cost of the card used to make 1000 boxes, all of a particular size, was $28. What does this suggest would be the cost of the same thickness and type of card used to make 5000 boxes each with side lengths twice the length of the corresponding sides of the first mentioned box.

6. A map has the scale 1 : 50000 which means 1 unit of length on the map is equivalent to 50000 of those units of length in real life. For example a distance of 1 cm on the map represents 50000 cm (= 500 metres) in real life.
An area of parkland occupies 4·5 cm^2 on the map. What is the area of this parkland in real life, in square kilometres?

7. The solid sphere of radius 1 metre is made of a particular metal. The metal contained in the sphere is worth $70000. What does this suggest the same metal forming a solid sphere or radius 2 metres would be worth?

8. A company making mining shovels, a machine used for digging and loading earth or rock on a mine site, makes two types, *The Little Joe* and *The Big John*. Each *Big John* is simply a scaled up version of a *Little Joe* with all lengths on the *Big John* being 1·2 times the equivalent length on a *Little Joe*. The bucket on a Little Joe has a capacity of 50 m^3. What is the capacity of the bucket on a Big John?

9. A map has a scale of 1 : 10000, i.e. 1 cm on the map represents 10000 cm on the ground. What area on the map will a real life area of 8000 m^2 occupy?

10. A model of a sailing boat is made to the scale 1 : 50, i.e a length of 1 cm on the model represents 50 centimetres on the real yacht. If one of the sails on the real yacht has an area of 5 m^2 what will be the area of the same sail on the model?

11. An object B is an enlargement of object A such that

    Volume of object B : Volume of object A = 27 : 8.

    What would the following ratios be
    (a) Length measured on object B : Corresponding length measured on object A.
    (b) Surface area of object B : Surface area of object A.

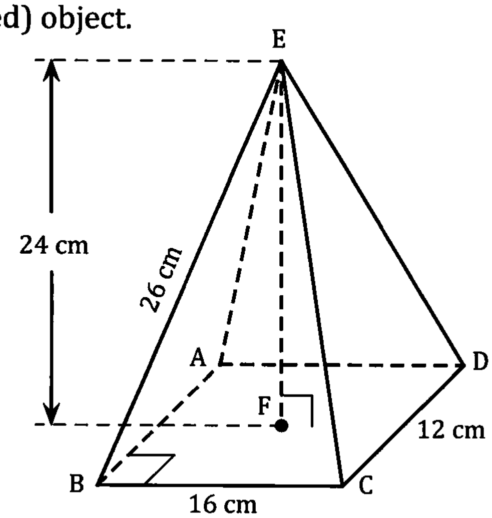
12. An object C is an enlargement of object D such that

    Surface area of object C : Surface area of object D = 16 :25.

    What would the following ratios be
    (a) Length measured on object C : Corresponding length measured on object D.
    (b) Volume of object C : Volume of object D.

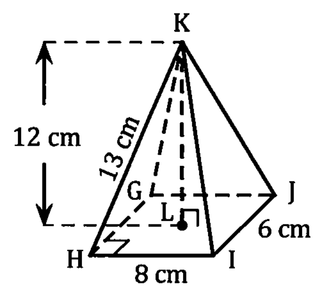
Similarity

If one 3-dimensional object is simply a scaled enlargement, or reduction, of another 3-dimensional object then the two objects are said to be **similar**. The ratio of any length on one of the objects to that of the corresponding length on the other object will be the same for all such pairs of lengths. Furthermore, any angle measured on one of the two objects will be the same size as the equivalent angle measure on the enlarged (or reduced) object.



Corresponding lengths in same ratio

BE : HK = 2 : 1
BC : HI = 2 : 1
CD : IJ = 2 : 1
Etc



Corresponding angles equal.

∠ABC = ∠GHI (= 90°)
∠EFB = ∠KLH (= 90°)
∠EBC = ∠KHI
Etc

Note also that volume of pyramid ABCDE : volume of pyramid GHIJK

$$= \frac{16\text{ cm} \times 12\text{ cm} \times 24\text{ cm}}{3} : \frac{8\text{ cm} \times 6\text{ cm} \times 12\text{ cm}}{3}$$

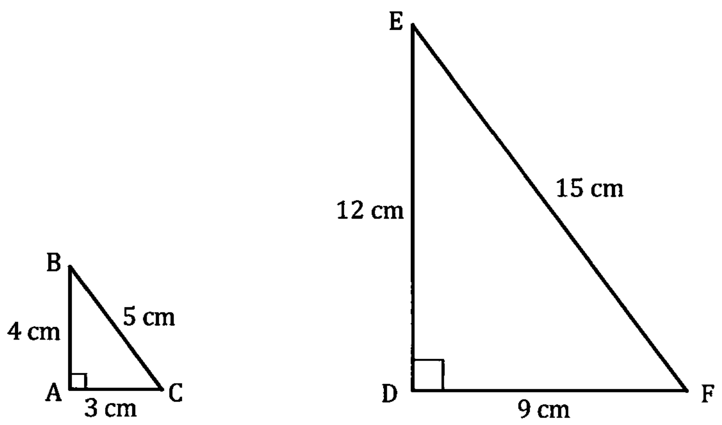
$$= 1536\text{ cm}^3 : 192\text{ cm}^3$$

$$= 8 : 1$$

$$\text{i.e. } 2^3 : 1$$

as we would expect from the earlier work of this chapter.

In the same way, two 2-dimensional shapes for which one is a scaled enlargement or reduction of the other are said to be **similar**. Again corresponding sides will be in the same ratio and corresponding angles will be equal.



Corresponding lengths in same ratio	Corresponding angles equal.
AB : DE = 1 : 3	$\angle$BAC = $\angle$EDF (= 90°)
AC : DF = 1 : 3	$\angle$ABC = $\angle$DEF
BC : EF = 1 : 3	$\angle$BCA = $\angle$EFD

Triangle ABC is similar to triangle DEF.
We write: $\Delta ABC \sim \Delta DEF$

The order of the letters is important. It would not be correct to say that $\Delta ABC \sim \Delta EFD$ for example. The corresponding vertices, A and D, B and E, C and F, should appear in corresponding places in the similarity statement.

Note also that for the two triangles shown above, that whilst

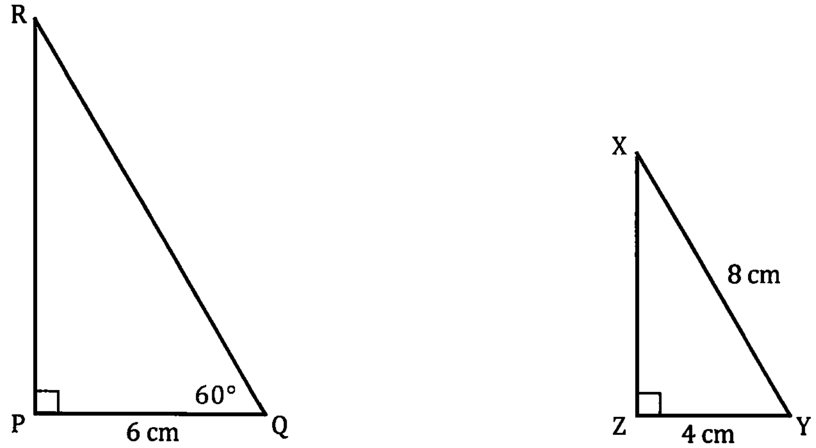
Length in ΔABC : Corresponding lengths in ΔDEF = 1 : 3,
Area of ΔABC : Area of ΔABC = 6 cm^2 : 54 cm^2
= 1 : 9
= 1 : 3^2

as we would expect from earlier work in this chapter.

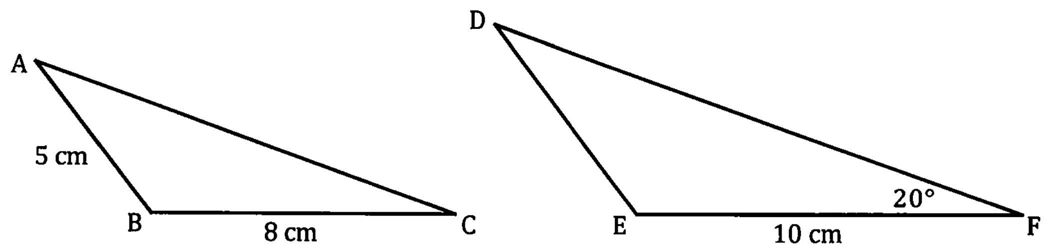
Note: An enlargement may also be referred to as a **dilation**. The medical world, for example, uses the terms vasodilation,
pupillary dilation,
and cervical dilation.

Exercise 10B.

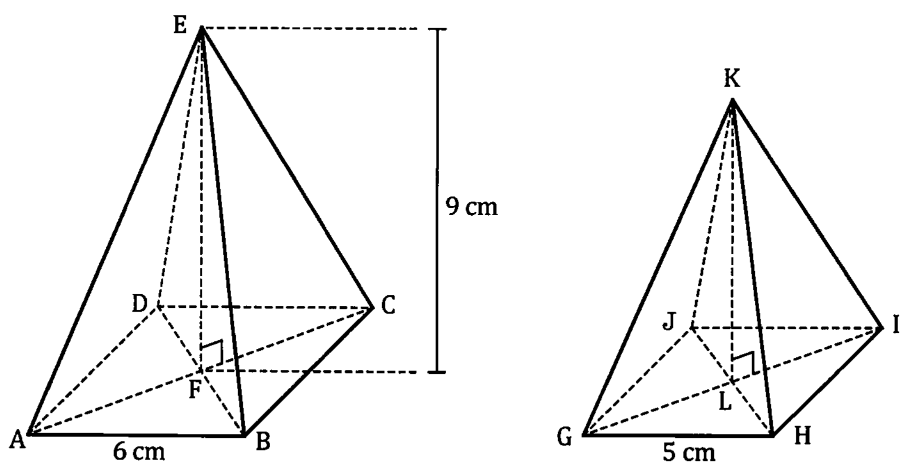
1. Given that triangles PQR and ZYX shown below are similar determine
   (a) the length of RQ, (b) the size of $\angle ZXY$,
   (c) the ratio area of triangle PQR : area of triangle ZYX.



2. Given that in the diagram below $\Delta ABC \sim \Delta DEF$ determine
   (a) the length of DE, (b) the size of $\angle ACB$,
   (c) the ratio area of triangle ABC : area of triangle DEF.

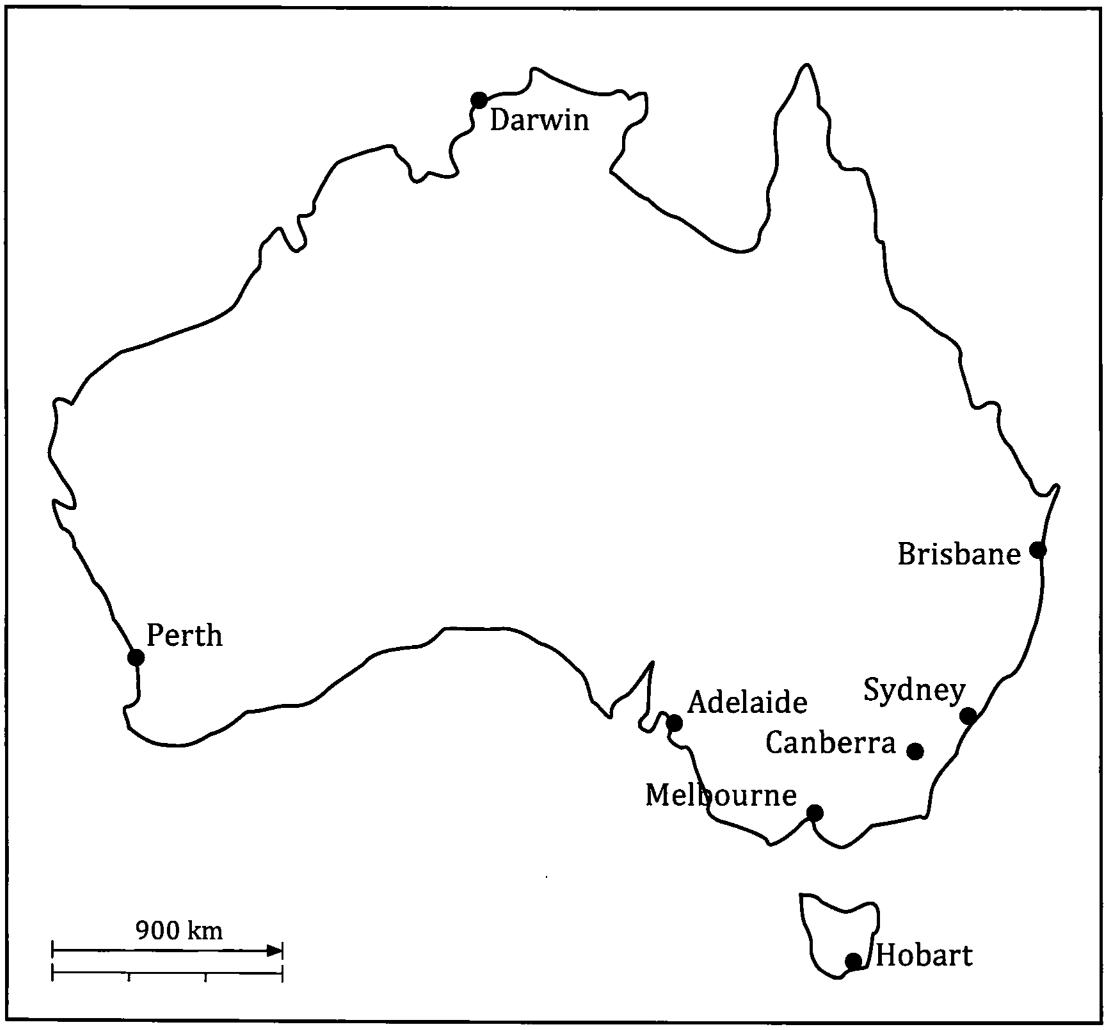


3. Given that the two square based pyramids shown below are similar determine
   (a) the length of KL.
   (b) the ratio surface area of pyramid ABCDE : surface area of pyramid GHIJK.
   (c) the ratio volume of pyramid ABCDE : volume of pyramid GHIJK.



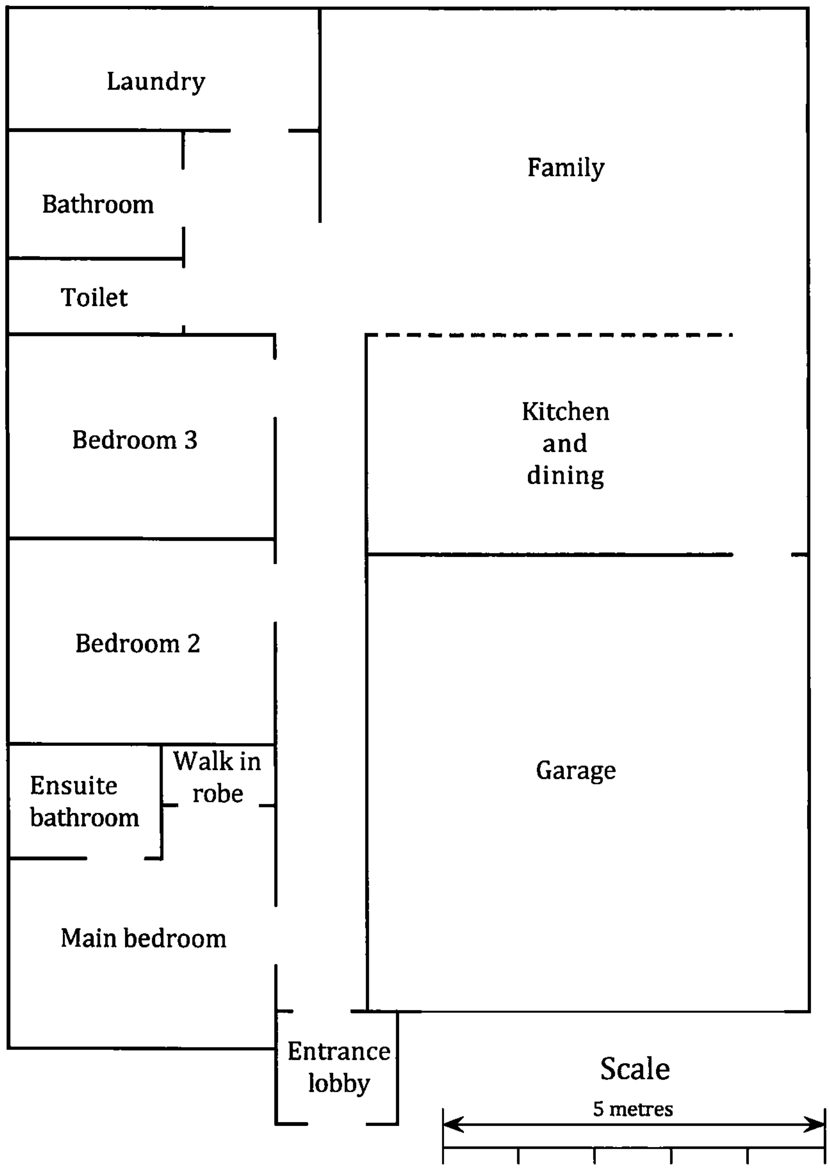
4. Measure distances on the map of Australia shown below to obtain approximate answers for the straight line distances from

   (a) Perth to Sydney,
   (b) Perth to Darwin,
   (c) Melbourne to Sydney,
   (d) Adelaide to Darwin,
   (e) Canberra to Brisbane.



   (f) On the map Sydney is approximately due east of Adelaide. Will this also be the case in real life?

5. The diagram below shows the floor plan of a house.



Measure appropriate distances to determine the dimensions of

(a) the garage,

(b) the main bedroom (including ensuite and walk in robe),

(c) bedroom 3,

(d) the laundry.

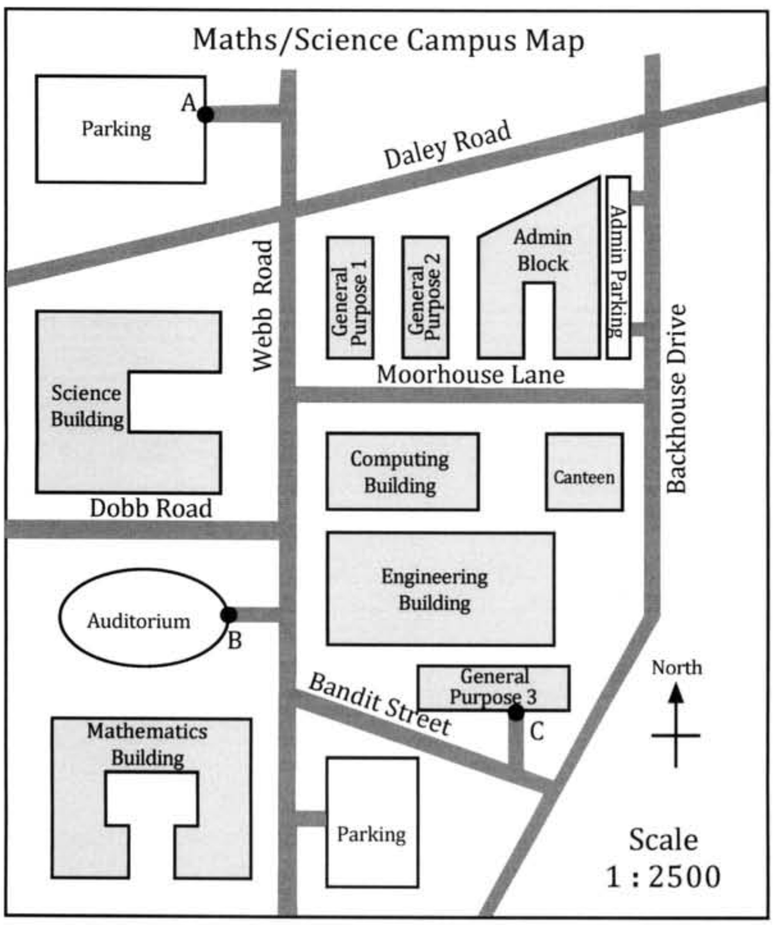
6. The map shown below shows the layout of the Mathematics and Science section of a University Campus.
Make appropriate measurements to determine each of the following.

(a) How far is it in real life from the point marked A, situated at the exit to the Northern car park, to the point marked B, the entrance to the Auditorium, travelling along the road sections by the most obvious route?

(b) How far is it in real life from the point marked A, situated at the exit to the Northern carpark, to the point marked C, the entrance to building "General Purpose 3", travelling the road route that involves Webb Road and Bandit Street.

(c) What is the area in real life of the "footprint" of the Computing building?

(d) What is the area in real life of the "footprint" of the Science building.

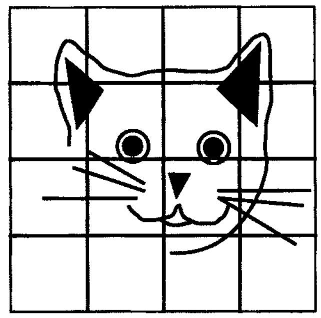


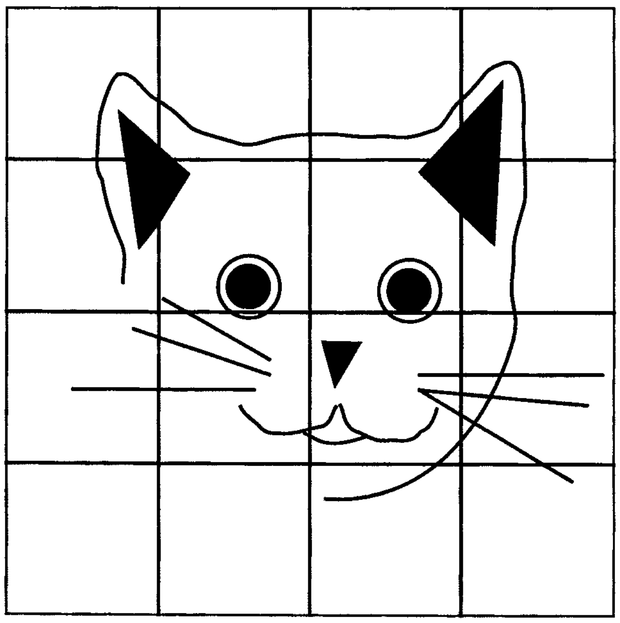
Drawing enlargements (and reductions).

Nowadays, with so many of the two dimensional diagrams we create being drawn on a computer it can be an easy task to make an enlargement or reduction of the drawing by "dragging" one corner of the diagram appropriately. (However, if we do want an enlargement of a diagram we must be careful that this "dragging" doesn't just stretch the diagram either horizontally of vertically, producing a "stretch" rather than a true enlargement.)

Alternatively a scaled drawing could be constructed using grid paper

If we have a picture drawn on grid paper, as shown on the right, then by carefully redrawing what is in each square, but on a larger grid, an enlargement can be produced, as shown below.





Conditions for similarity.

When we create an enlargement or reduction of a shape the enlargement or reduction is *similar* to the original because:

☞ all corresponding lengths are in the same ratio

and

☞ all corresponding angles are equal.

If instead we are given two shapes and are asked if they are similar we need to check that corresponding sides are in the same ratio and corresponding angles are equal. If both of these conditions are met then the two shapes are similar.

Notice that in general we cannot assume similarity if just one of the conditions

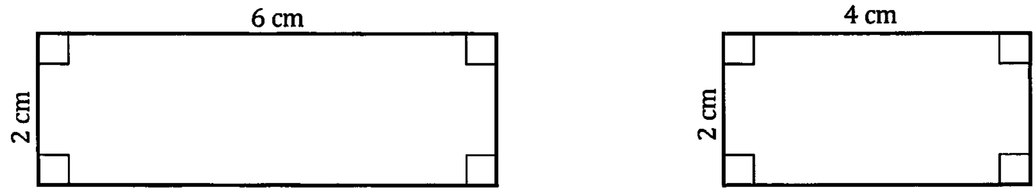
corresponding sides in the same ratio

corresponding angles are equal

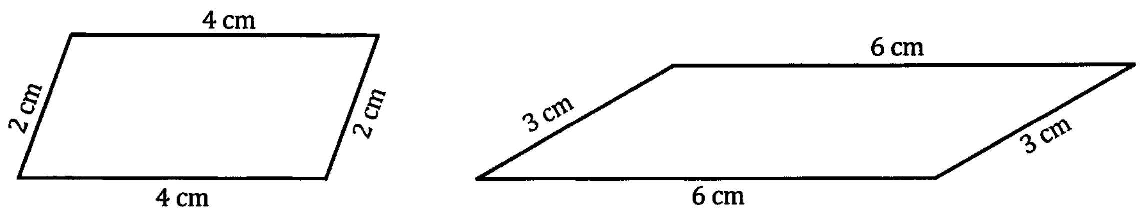
hold. We need both of these conditions to be the case for similarity.

To illustrate this consider the following:

The two rectangles shown below may have their angles matching but the rectangles are not similar.



The parallelograms shown below may have sides in the same ratio but the parallelograms are not similar.

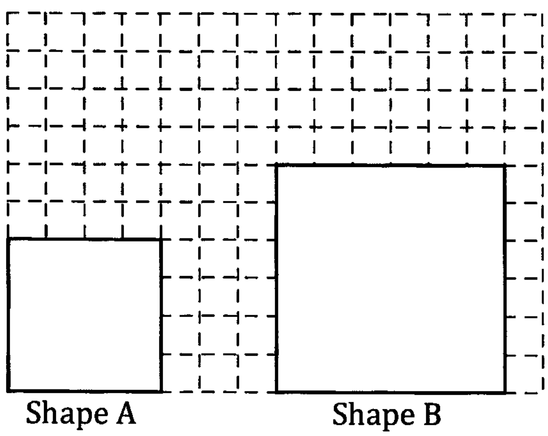


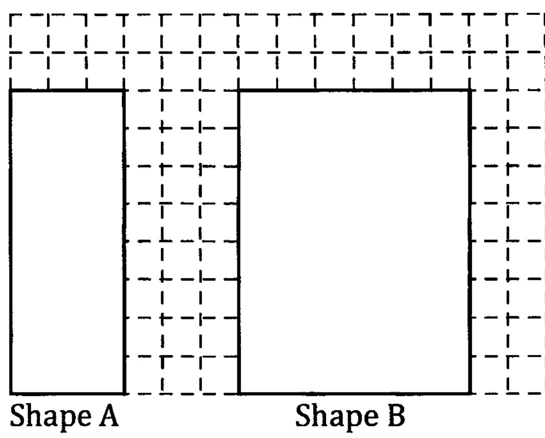
Note: The statement at the top of this page did say "in general". For triangles the situation is a little different, as we shall see later in this chapter.

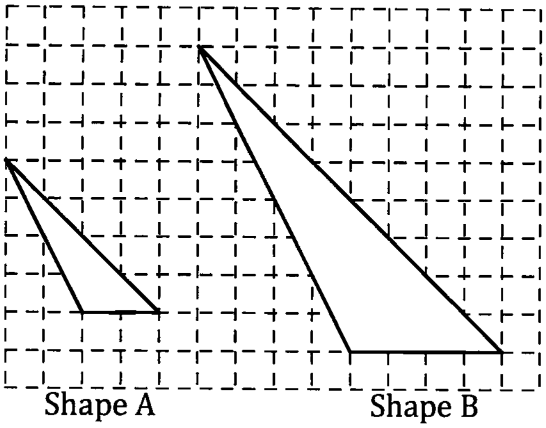
Exercise 10C.

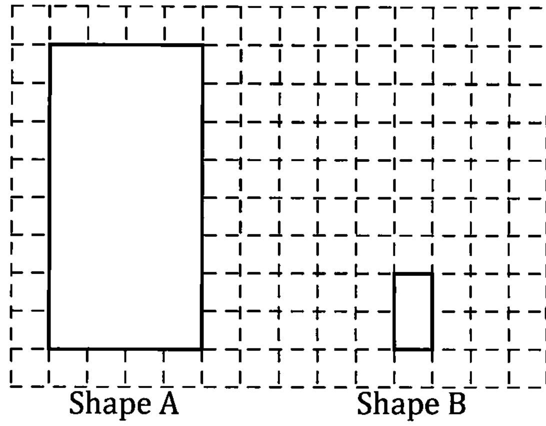
For each of questions 1 to 8, state whether or not the two shapes shown are *similar* or *not similar* and for those that are state the ratio

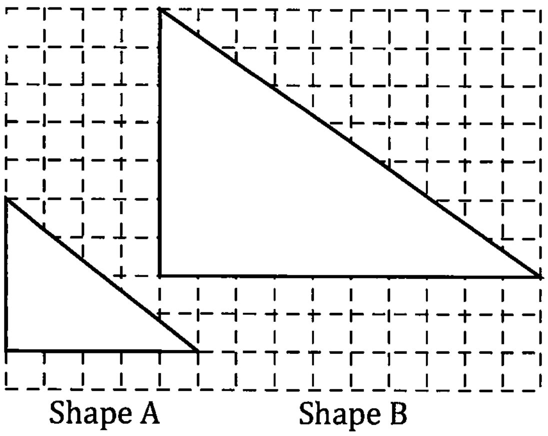
Lengths in shape A : corresponding lengths in shape B.

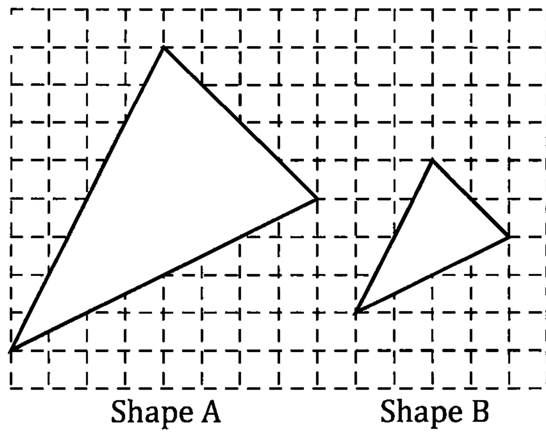
1. 

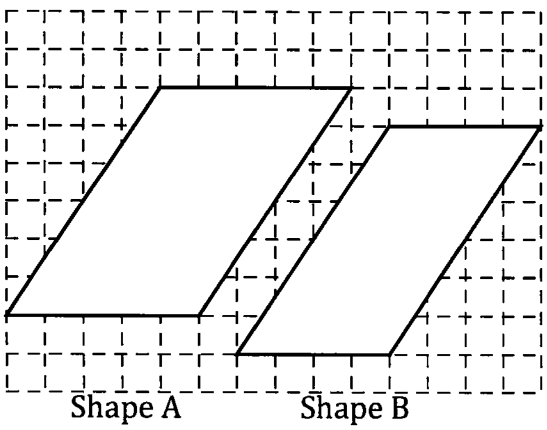
2. 

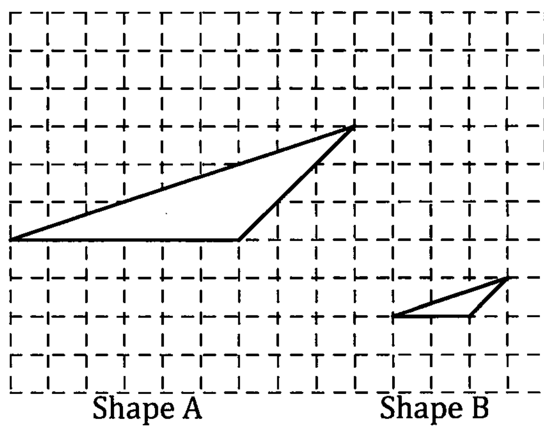
3.


4.


5.


6.


7.


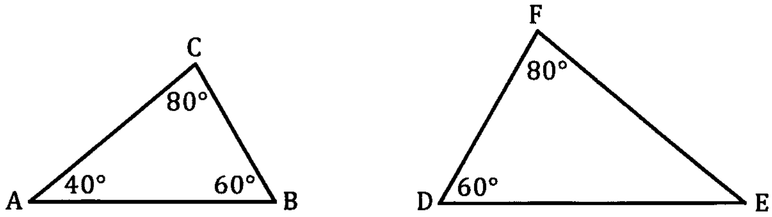
8.


9. Are all triangles similar to each other?
10. Are all rectangles similar to each other?
11. Are all squares similar to each other?
12. Are all isosceles triangles similar to each other?
13. Are all right angled triangles similar to each other?
14. Are all equilateral triangles similar to each other?

Similar triangles.

As was mentioned earlier, triangles do <u>not</u> fit the more general condition that to know that two shapes are similar we must check <u>both</u> that corresponding sides are in the same ratio <u>and</u> that corresponding angles are equal. Indeed to know whether two triangles are similar we can:

- See if the three angles of one triangle are equal to the three angles of the other triangle.

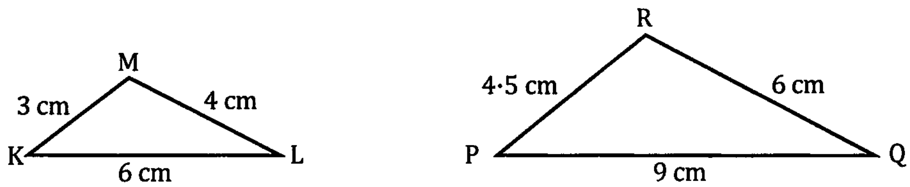


Noticing that each triangle has angles of 40°, 60° and 80° we can say that ΔABC and ΔEDF are similar. (Note carefully the order of the letters. The second triangle is listed as "EDF" to match the corresponding angles in "ABC".)

We write: ΔABC ~ ΔEDF Reason: Corresponding angles equal.

OR:

- See if the lengths of corresponding sides are in the same ratio.



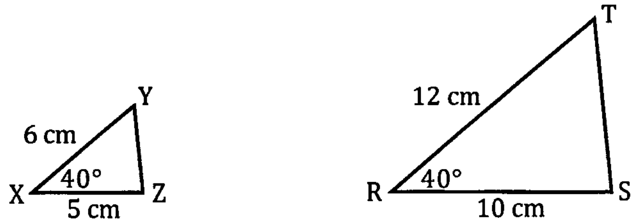
KL : PQ = 6 : 9	LM : QR = 4 : 6	KM : PR = 3 : 4·5
= 2 : 3	= 2 : 3	= 2 : 3

Thus ΔKLM and ΔPQR are similar.

We write: ΔKLM ~ ΔPQR Reason: Corresponding sides in same ratio.

OR:

- See if the lengths of two pairs of corresponding sides are in the same ratio and the angles between the sides are equal.



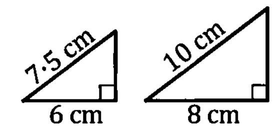
XZ : RS = 5 : 10	XY : RT = 6 : 12
= 1 : 2	= 1 : 2

The angle between XY and XZ = the angle between RT and RS.

Hence ΔXZY ~ ΔRST Reason: Two pairs of corresponding sides in same ratio and the *included* angles equal.

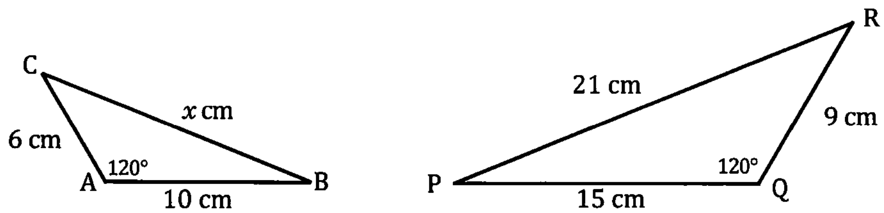
Note: ① The fact that the angles of a triangle have a sum of 180° means that once we have shown that two angles of one triangle are equal in size to two angles in another triangle the third angles must be equal. The condition that corresponding angles are equal is then satisfied and the triangles are similar.

② If the two triangles are right angled the corresponding sides that are in the same ratio need not *include* the right angle.



Example 1

Explain why the two triangles shown below are similar and hence find x.



$$AB : QP = 10 : 15 = 2 : 3$$

$$AC : QR = 6 : 9 = 2 : 3$$

$$\angle CAB = \angle RQP \quad (= 120°)$$

Thus $\Delta ABC \sim \Delta QPR$ Reason: Two pairs of corresponding sides in same ratio and the included angles equal.

Hence $CB : RP = 2 : 3$

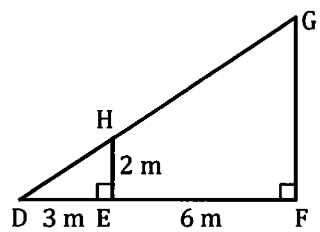
$x : 21 = 2 : 3$

but $14 : 21 = 2 : 3$

Hence $x = 14$

Example 2

In the diagram shown on the right DE = 3 m, EF = 6 m, and EH = 2 m. Find the length of FG, justifying your answer.



In triangles DEH and DFG: $\angle HDE = \angle GDF$ (same angle)

$\angle HED = \angle GFD$ (= 90°)

Hence the third angles will be equal and so $\Delta DEH \sim \Delta DFG$, corresponding angles equal.

Hence $DE : DF = EH : FG$

Letting the length of FG be x cm:

$3 : 9 = 2 : x$

i.e. $1 : 3 = 2 : x$

But $1 : 3 = 2 : 6$

Hence $x = 6$

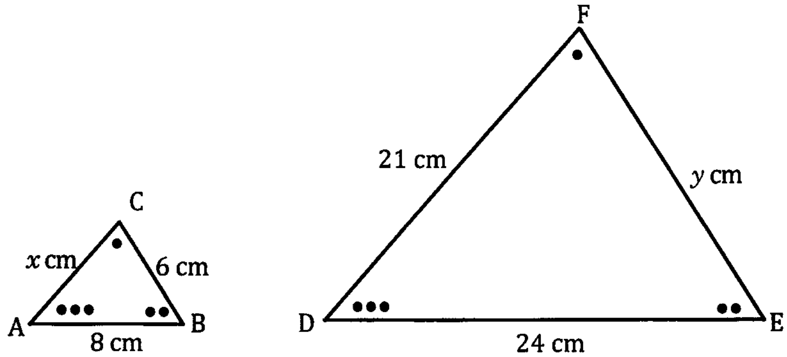
FG is of length 6 cm.

Exercise 10D.

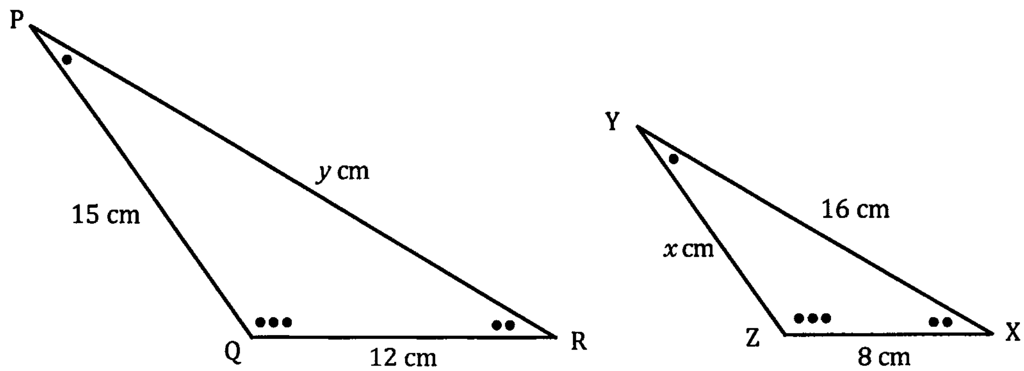
For questions 1 to 8 of this exercise state whether the two triangles are similar or not and, **for those that are**, name the similar triangles, explain why they are similar, state the ratio of their areas and determine the values of x and y as appropriate.

☞ Within each question any angles marked • are the same size as each other,
any angles marked •• are the same size as each other,
any angles marked ••• are the same size as each other,

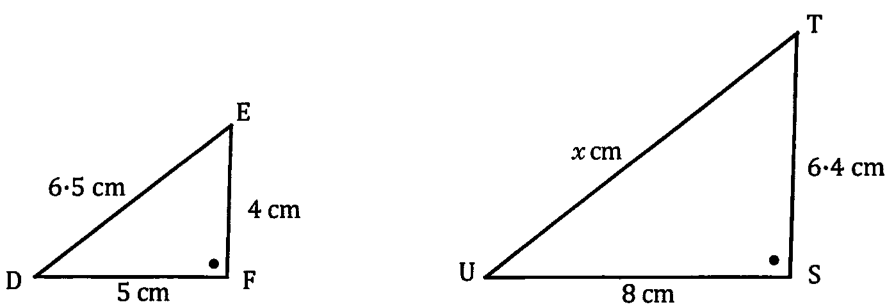
1.



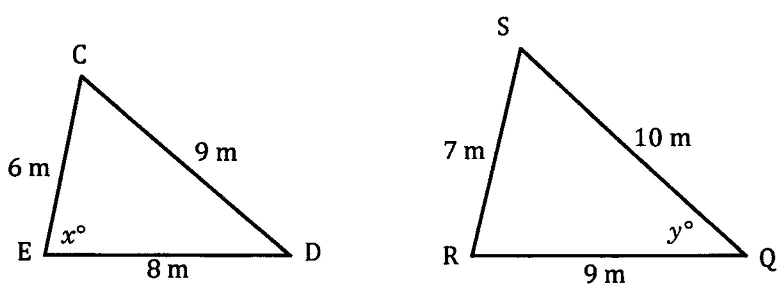
2.



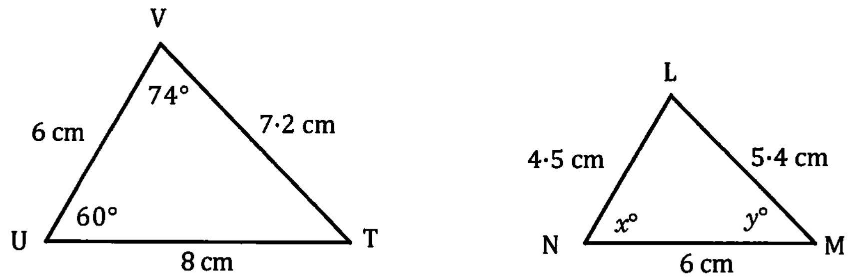
3.



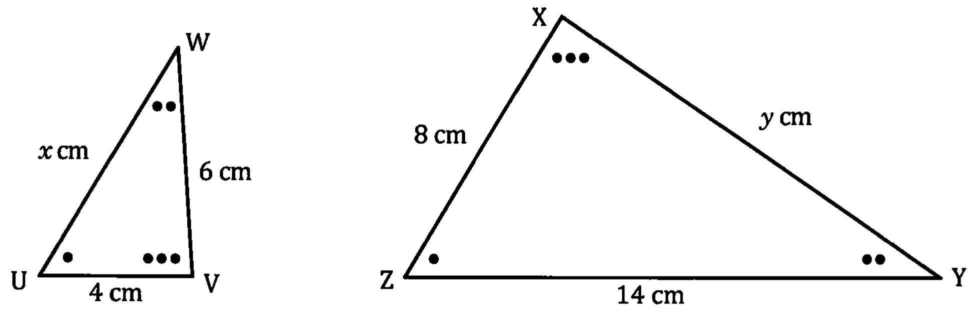
4.



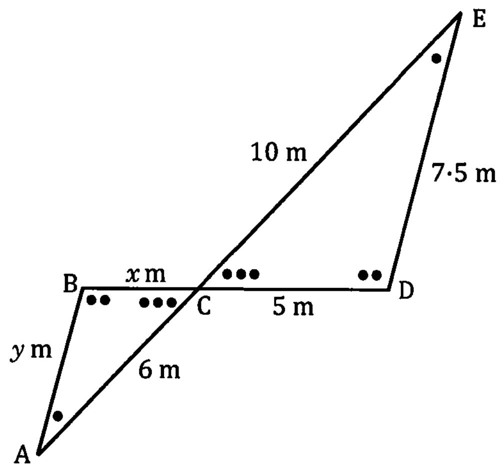
5.



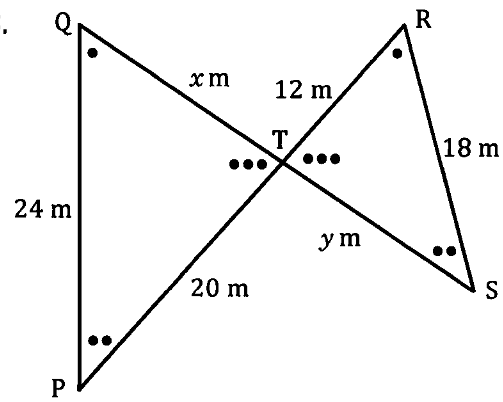
6.



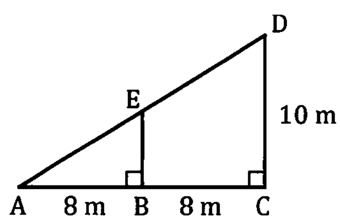
7.



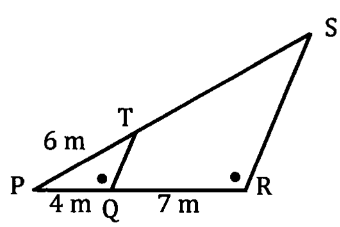
8.



9. Determine the length of EB given the information in the diagram shown on the right. (Justify your answer.)



10. Determine the length of TS given the information in the diagram shown on the right. (Justify your answer.)



Miscellaneous Exercise Ten.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary section at the beginning of the book.

1. When evaluating πr^2 we multiply a number, π, by a length, r, and then by a length, r, again. Multiplying two lengths together give units of area. Multiplying by π, which is just a number and has no units, will not change the units of the answer.
   Thus even if we did not recognise πr^2 as the formula for the area of a circle of radius r we could still identify it as a formula that will give an answer involving units of area.
   In each of the following a, b, c, h and r all represent lengths and any numbers and π are "unitless". For each part state whether the answer would be a length, an area or a volume.

   (a) $a + b$ (b) $a + b + c$ (c) c^2

   (d) ab (e) bc (f) $2a + b$

   (g) abc (h) $\frac{bh}{2}$ (i) $3c^2$

   (j) $\frac{(a + b)}{2}h$ (k) $4\pi r^2$ (l) $2\pi r$

   (m) $\frac{\pi r^2 h}{3}$ (n) $\frac{4}{3}\pi r^3$ (o) $\frac{1}{2}abh$

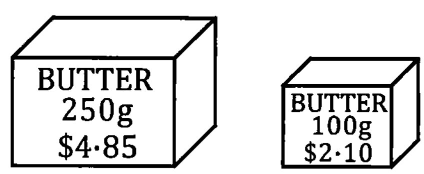
2. Express each of the following numbers in scientific notation (i.e. express each in the form $A \times 10^n$ where A is a number between 1 and 10 and n is an integer).

   (a) 1 230 000 (b) 0·001 2
   (c) 25 000 (d) 0·000 000 024 5
   (e) 15 000 000 000 (f) 0·000 03
   (g) 76 (h) 0·1

3. A triangle has sides of length 5 cm, 8 cm and 10 cm.
   A second triangle, that is similar to the first, has a perimeter of 92 cm. What are the side lengths of this second triangle?

4. Tony calculates that he will need 5 bags of fertlliser to fertilise his front lawn which is rectangular in shape and has dimensions 5 metres × 8 metres.
   How much fertilizer would be needed to fertilise a rectangular lawn that is three times as long and three times as wide as Tony's, assuming the same rate of application?

5. Determine the better buy for the butter deals shown on the right.



6. A map has a scale of 1 : 50 000, i.e. 1 cm on the map represents 50 000 cm (= 500 metres) on the ground.
   (a) On the map the straight line distance between two locations is 154 mm. How far apart are these two locations in real life?
   (b) If two locations are 2 km apart in real life, how far apart are they on the map?
   (c) On the map the grounds of a shopping centre occupies an area of 1 cm^2. What area does this shopping centre occupy in real lif?

7. What is the ratio of the lengths of corresponding sides in two similar triangles given that the ratio of the areas of these two triangles is 49 : 25 ?

8. Which earns more interest and by how much more:

   $2000 invested for 4 years at 8% per annum compounded six monthly

   or

   $1800 invested for 5 years at 6% per annum compounded monthly.

9. For the matrices $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ form all of the products DE that can be formed given that D can be chosen from A, B or C and E can be chosen from A, B or C.

10. A reduced size scale model of a vehicle is made to a scale of 2 : 35. If the real vehicle is of length 3·85 metres how long is the scale model?

11. Three shops each have special deals on a particular brand of chocolate.

    Shop A: Buy two 250 gram bars, normally $4·20 each, and we will give you 80 cents off each bar.

    Shop B: Buy ten 200 gram bars at $2·95 each bar and we will give you two extra 200 gram bars free.

    Shop C: 15% off the normal cost of a 500 gram bar. (Normal cost $7·80.)

    Rank the three deals, best value for money first.

12. A map has a scale of 1 : 500 000.
    (a) The distance between two locations on the map is 6·2 cm. How far are these two locations apart on the ground?
    (b) A forestry area occupies 0·8 cm^2 on the map. What area does this forestry area occupy in real life?

13. A solid metal cube of side length 10 cm is melted down and the metal is recast into cubes of side length 1 cm.

    Assuming no metal is lost in this process:
    (a) How many of the smaller cubes will there be?
    (b) How does the total surface area of all the smaller cubes compare with the surface area of the original cube?

14. Let us suppose that a particular government child education allowance gives, to each family, \$250 per month per child in full time education and living at home. However the total monthly amount received under this scheme reduces by \$0·50 for each dollar that the combined parental earned income exceeds \$850 per week. For what combined parental weekly earned income does this allowance cut out (i.e. reduce to zero) for a family with (a) 1 child, (b) 2 children, (c) 3 children.

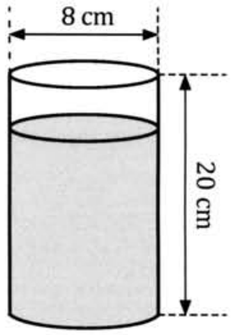
15. How many rectangular blocks with dimensions a cm × b cm × c cm can fit into a rectangular space that is $5a$ cm × $5b$ cm × $5c$ cm?

16. If it takes 16 seconds for a machine to pump up a balloon of radius 30 cm how long does this suggest it would take the machine to pump up a balloon of radius 45 cm?

17. Two spheres, A and B, are such that the ratio of the surface area of sphere A to that of sphere B is 1 : 16. If sphere A has a volume of 50 cm^3 what is the volume of sphere B?

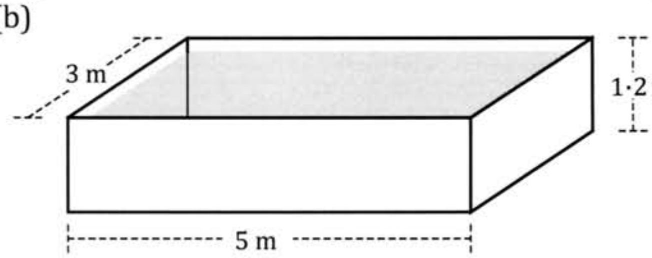
18. How much sodium chloride should be added to each of the following containers, assumed 80% full of water, to give a concentration of 3 grams of sodium chloride per litre of water. (Assume that the given measurements are internal dimensions.)

(a)



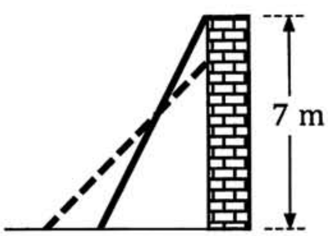
(A cylindrical container.)

(b)



(A rectangular pool of uniform depth.)

19. An 8 metre ladder is placed with its foot on horizontal ground and its top just reaching the top of a vertical wall of height 7 metres. With the ladder remaining in contact with both the wall and the ground, the base of the ladder is pulled a further 1 metre from the wall. How far does this action cause the top of the ladder to move down the wall?



20. Jen works at a garden centre and is given the job of polishing the two large solid stone spheres that proudly adorn the entrance to the centre, one of the spheres being twice the volume of the other.
Jen starts on the smaller one and finds that it takes her 10 minutes to polish it.
How long would this suggest it would take Jen to polish the larger one?
Explain any assumptions you make in calculating your answer.