NELSON QMATHS 11 MATHEMATICAL METHODS

FULLY WORKED SOLUTIONS

Chapter 1 Arithmetic sequences and series

Exercise 1.01 Sequences and recursive definitions

Question 1

a 13, 15, 17, 19

b $t_9 = 19,\ t_{10} = 21,\ t_{11} = 23,\ t_{12} = 25$

c $t_{13} = 27,\ t_{14} = 29,\ \text{so}\ t_{14}$

Question 2

a Add 4: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44

$\therefore\ t_{11} = 44$

b Subtract 5: 3, –2, –7, –12, –17, –22, –27, –32, –37, –42

$\therefore t_{10} = -42$

c Odd numbers squared, or add 8, 16, 24, etc: 1, 9, 25, 49, 81, 121, 169, 225

$\therefore t_8 = 225$

d Multiply by 4 4: 3, 12, 48, 192, 768, 3072, 12 288

$\therefore t_7 = 12\,288$

Question 3

a Add 2:

$t_5 = 13,\ t_6 = 15,\ t_7 = 17,\ t_8 = 19,\ t_9 = 21,\ t_{10} = 23,\ t_{11} = 25,\ t_{12} = 27,\ t_{13} = 29,\ t_{14} = 31,$

$t_{15} = 33,\ t_{16} = 35,\ t_{17} = 37,\ t_{18} = 39,\ t_{19} = 41,\ t_{20} = 43,\ t_{21} = 45,\ t_{22} = 47,\ t_{23} = 49$

$\therefore t_{23}$

b Subtract 7: $t_5 = -16,\ t_6 = -23,\ t_7 = -30,\ t_8 = -37,\ t_9 = -44,\ t_{10} = -51,\ t_{11} = -58$

$\therefore t_{11}$

c Multiply by 3: $t_5 = 81,\ t_6 = 243,\ t_7 = 729,\ t_8 = 2187$

$\therefore t_8$

d Divide by 5 or multiply by 0.2:

$t_5 = 0.0004,\ t_6 = 0.00008,\ t_7 = 0.000016$

$\therefore t_7$

Question 4

a $t_1 = 5$

$t_2 = 5 + 3 = 8$
$t_3 = 8 + 3 = 11$
$t_4 = 11 + 3 = 14$
$t_5 = 14 + 3 = 17$

$\therefore$ Sequence is 5, 8, 11, 14, 17, …

b $t_1 = 5 = 3\times 1 + 2$

$t_2 = 5 + 3 = 8 = 3\times 2 + 2$
$t_3 = 8 + 3 = 11 = 3\times 3 + 2$
$t_4 = 11 + 3 = 14 = 3\times 4 + 2$
$t_5 = 14 + 3 = 17 = 3\times 5 + 2$

$\therefore t_n = 3n + 2$

Question 5

a $t_1 = 12$

$t_2 = 12 - 4 = 8$

$t_3 = 8 - 4 = 4$

$t_4 = 4 - 4 = 0$

$t_5 = 0 - 4 = -4$

$\therefore$ First 5 terms are: 12, 8, 4, 0, –4

b $t_1 = 12$

$t_2 = 12 \div 2 = 6$

$t_3 = 6 \div 2 = 3$

$t_4 = 3 \div 2 = 1.5$

$t_5 = 1.5 \div 2 = 0.75$

$\therefore$ First 5 terms are: 12, 6, 3, 1.5,

0.75

c $t_1 = 4$

$t_2 = 2 \times 4 - 1 = 7$

$t_3 = 2 \times 7 - 1 = 13$

$t_4 = 2 \times 13 - 1 = 25$

$t_5 = 2 \times 25 - 1 = 49$

$\therefore$ Sequence is 4, 7, 13, 25, 49

d $t_1 = 6$

$t_2 = \dfrac{6+4}{2} = 5$

$t_3 = \dfrac{5+4}{2} = 4.5$

$t_4 = \dfrac{4.5+4}{2} = 4.25$

$t_5 = \dfrac{4.25+4}{2} = 4.125$

$\therefore$ Sequence is 6, 5, 4.5, 4.25,

4.125

e $t_1 = \dfrac{3}{16}$

$t_2 = 4 \times \dfrac{3}{16} = \dfrac{3}{4}$

$t_3 = 4 \times \dfrac{3}{4} = 3$

$t_4 = 4 \times 3 = 12$

$t_5 = 4 \times 12 = 48$

$\therefore$ Sequence is $\dfrac{3}{16}$, $\dfrac{3}{4}$, 3, 12, 48

Question 6

a $t_1 = 7$

$t_2 = 7 + 3 = 10$

$t_3 = 10 + 3 = 13$

$t_4 = 13 + 3 = 16$

$t_5 = 16 + 3 = 19$

$\therefore t_n = 3n + 4$

b $t_1 = 80$

$t_2 = 20$

$t_3 = 5$

$t_4 = 1.25$

$\therefore\ t_n = \dfrac{80}{4^{n-1}}$ or $t_n = 80 \times 4^{1-n}$

c $t_1 = 8$

$t_2 = \dfrac{2 \times 8 + 5}{2} = 10.5$

$t_3 = \dfrac{2 \times 10.5 + 5}{2} = 13$

$t_4 = \dfrac{2 \times 13 + 5}{2} = 15.5$

$t_5 = \dfrac{2 \times 15.5 + 5}{2} = 18$

$\therefore\ t_n = 2.5n + 5.5$

d $t_1 = 1$

$t_2 = 8$

$t_3 = 27$

$t_4 = 64$

$\therefore\ t_n = n^3$

e $t_1 = 5$

$t_2 = \dfrac{5 + 4}{2} = 4.5$

$t_3 = \dfrac{4.5 + 4}{2} = 4.25$

$t_4 = \dfrac{4.25 + 4}{2} = 4.125$

$\therefore\ t_n = 4 + \left(\dfrac{1}{2}\right)^{n-1}$ or $4 + 2^{1-n}$

Question 7

a Sequence is 240, 260, 280, …

$\therefore t_n = 240 + 20n$

$\therefore t_8 = 240 + 20 \times 8 = 400$

$\therefore t_{10} = 240 + 20 \times 10 = 440$

$\therefore t_{20} = 240 + 20 \times 20 = 640$

$\therefore$ In 8 weeks she will have \$400. In 10 weeks she will have \$440.
In 20 weeks she will have \$640.

b $800 = 240 + 20n$

$20n = 560$

$\therefore n = 28$

$\therefore$ It will take her 28 weeks.

Question 8

a Sequence is 24, 20, 18, 17, 16.5, 16.25, 16.125, …

$\therefore$ In 3 weeks time, will be $16\frac{1}{8}$ s.

b Sequence will continue as 16.125, 16.062 5, 16.031 25, …

$\therefore$ As the difference in his times is improving by only half of the improvement he showed in his previous run, the times will approach 16 but never go below 16 seconds. Hence, the best time he can do is 16 seconds.

Exercise 1.02 Arithmetic sequences

Question 1

a $t_2 - t_1 = -2,\ t_3 - t_2 = -2,\ t_4 - t_3 = -2$

There is a common difference of –2 for successive terms, so it is an AP.

b $t_2 - t_1 = 1 - 2x, t_3 - t_2 = 1 - 2x, t_4 - t_3 = 1 - 2x$

There is a common difference of 1 – 2x for successive terms, so it is an AP.

c $t_2 - t_1 = 0.2, t_3 - t_2 = 0.02, t_4 - t_3 = 0.002$

The difference of successive terms is not a constant, so it is not an arithmetic sequence.

d $t_2 - t_1 = 2, t_3 - t_2 = 3, t_4 - t_3 = 4$

The difference of successive terms is not a constant, so it is not an arithmetic sequence.

e $t_2 - t_1 = \frac{1}{12}, t_3 - t_2 = \frac{1}{12}, t_4 - t_3 = \frac{1}{12}$

There is a common difference of difference of $\frac{1}{12}$ for successive terms, so it is an AP.

f $t_2 - t_1 = 6, t_3 - t_2 = 12, t_4 - t_3 = 24$

The difference of successive terms is not a constant, so it is not an arithmetic sequence.

Question 2

a

$t_2 - t_1 = 4,\ t_3 - t_2 = 4$

$\therefore\ a = 3,\ d = 4$

$t_n = a + (n-1)d$
$t_n = 3 + (n-1) \times 4$
$t_n = 4n - 1$

b

$t_2 - t_1 = -3,\ t_3 - t_2 = -3$

$\therefore\ a = 102,\ d = -3$

$t_n = a + (n-1)d$
$t_n = 102 + (n-1) \times -3$
$t_n = 105 - 3n$

c

$t_2 - t_1 = -\frac{7}{6},\ t_3 - t_2 = -\frac{7}{6}$

$\therefore\ a = 2\frac{1}{2},\ d = -\frac{7}{6}$

$t_n = a + (n-1)d$
$t_n = 2\frac{1}{2} + (n-1) \times -\frac{7}{6}$
$t_n = \frac{15 - 7n + 7}{6}$
$t_n = \frac{22 - 7n}{6}$

d

$t_2 - t_1 = 2.1,\ t_3 - t_2 = 2.1$

$\therefore\ a = 1.8,\ d = 2.1$

$t_n = a + (n-1)d$
$t_n = 1.8 + (n-1) \times 2.1$
$t_n = 2.1n - 0.3$

e

$t_2 - t_1 = 2,\ t_3 - t_2 = 2$

$\therefore\ a = -15,\ d = 2$

$t_n = a + (n-1)d$
$t_n = -15 + (n-1) \times 2$
$t_n = 2n - 17$

f

$t_2 - t_1 = 2x - y,\ t_3 - t_2 = 2x - y$

$\therefore a = 2x + 3y,\ d = 2x - y$

$t_n = a + (n-1)d$
$t_n = 2x + 3y + (n-1) \times (2x - y)$
$t_n = 2nx + (4-n)y$

Question 3

a $a = 2,\ d = -3$

$$t_n = 2 + (n-1) \times -3$$
$$= 5 - 3n$$
$$t_7 = 5 - 3 \times 7$$
$$= -16$$
$$t_{20} = 5 - 3 \times 20$$
$$= -55$$

b $a = 3,\ d = 8$

$$t_n = 3 + (n-1) \times 8$$
$$= 8n - 5$$
$$t_{11} = 8 \times 11 - 5$$
$$= 83$$
$$t_{16} = 8 \times 16 - 5$$
$$= 123$$

c $a = 3x, d = 2x - 2$

$$t_n = 3x + (n-1)(2x-2)$$
$$t_9 = 3x + 8(2x-2)$$
$$= 19x - 16$$
$$t_{32} = 3x + 31(2x-2)$$
$$= 65x - 62$$

d $a = -2\frac{1}{2}, d = \frac{3}{4}$

$$t_n = -2\frac{1}{2} + (n-1) \times \frac{3}{4}$$
$$= \frac{3n}{4} - \frac{13}{4}$$
$$t_6 = \frac{3 \times 6}{4} - \frac{13}{4}$$
$$= \frac{5}{4}$$
$$= 1\frac{1}{4}$$
$$t_{12} = \frac{3 \times 12}{4} - \frac{13}{4}$$
$$= \frac{23}{4}$$
$$= 5\frac{3}{4}$$

e $a = 0.\overline{4}, d = 0.\overline{2}$

$t_n = 0.\overline{4} + (n-1) \times 0.\overline{2}$

$= \frac{4}{9} + (n-1) \times \frac{2}{9}$

$= \frac{2}{9} + \frac{2n}{9}$

$= \frac{2(n+1)}{9}$

$t_9 = \frac{2 \times 10}{9}$

$= 2\frac{2}{9}$

$= 2.\overline{2}$

$t_{15} = \frac{2 \times 16}{9}$

$= 3\frac{5}{9}$

$= 3.\overline{5}$

f $a = 7, d = -6$

$t_n = 7 + (n-1) \times (-6)$

$= 13 - 6n$

$t_{20} = 13 - 6 \times 20$

$= -107$

$t_{30} = 13 - 6 \times 30$

$= -167$

Question 4

Let $t_1 = a$ and common difference = d.

$$
\begin{aligned}
a+11d &= 16 \\
a+19d &= 5 \\
(a+19d)-(a+11d) &= 16-5 \\
8d &= -11 \\
d &= -\frac{11}{8}
\end{aligned}
$$

$$a+11\times\left(-\frac{11}{8}\right)=16$$

$$a=\frac{249}{8}$$

$$
\begin{aligned}
t_{30} &= a+29d \\
&= \frac{249}{8}-29\times\frac{11}{8} \\
&= -\frac{70}{8} \\
&= -8\frac{3}{4}
\end{aligned}
$$

Question 5

Let $t_1 = a$ and common difference = d.

$$
\begin{aligned}
a+6d &= 2.9 \\
a+8d &= 4.7 \\
(a+8d)-(a+6d) &= 4.7-2.9 \\
2d &= 1.8 \\
d &= 0.9
\end{aligned}
$$

$$a+6\times 0.9=2.9$$

$$a=-2.5$$

$$
\begin{aligned}
t_{16} &= a+15d \\
&= -2.5+15\times 0.9 \\
&= -2.5+13.5 \\
&= 11
\end{aligned}
$$

Question 6

Let $t_1 = a$ and common difference = d.

$$a+3d=x$$
$$a+8d=y$$
$$(a+8d)-(a+3d)=y-x$$
$$5d=y-x$$
$$d=\frac{y-x}{5}$$

$$a+\frac{3(y-x)}{5}=x$$

$$a=x-\frac{3(y-x)}{5}$$
$$=\frac{8x-3y}{5}$$

$$t_{16}=a+29d$$
$$=\frac{8x-3y}{5}+15\times\frac{(y-x)}{5}$$
$$=\frac{12y-7x}{5}$$

Question 7

$a = 3$, $d = 4.1$, $n = ?$

$$200=3+(n-1)4.1$$
$$=3+4.1n-4.1$$
$$=4.1n-1.1$$
$$201.1=4.1n$$
$$n=\frac{201.1}{4.1}=49.04...$$

50th term

(Round up to the nearest whole number if the sum of the sequence is to exceed 200).

Question 8

$t_5 = a + 4d$

$t_{10} = a + 9d$

$t_{20} = a + 19d = 80$

$$
\begin{aligned}
(a + 4d) + (a + 9d) &= 80 \\
2a + 13d &= 80 \\
a + 19d &= 80
\end{aligned}
$$

Solve above equations simultaneously.

$$
\begin{aligned}
2a + 13d &= 80 \\
2a + 38d &= 160 \\
(2a + 38d) - (2a + 13d) &= 160 - 80 \\
25d &= 80 \\
d &= 3.2
\end{aligned}
$$

$$
\begin{aligned}
a + 19 \times 3.2 &= 80 \\
a &= 19.2
\end{aligned}
$$

$$
\begin{aligned}
t_{30} &= 19.2 + 3.2 \times 29 \\
&= 112
\end{aligned}
$$

Question 9

$t_p = x \quad \therefore a + (p-1)d = x \quad [1]$

$t_q = y \quad \therefore a + (q-1)d = y \quad [2]$

[1] – [2]:

$$
\begin{aligned}
a + (p-1)d - a - (q-1)d &= x - y \\
(p-1)d - (q-1)d &= x - y \\
pd - d - qd + d &= x - y \\
d(p-q) &= x - y \\
d &= \frac{x-y}{p-q}
\end{aligned}
$$

Substitute into [1]:

$$a + (p-1)\left(\frac{x-y}{p-q}\right) = x$$

$$
\begin{aligned}
a &= x - \frac{(p-1)(x-y)}{p-q} \\
&= \frac{x(p-q) - (px - py - x + y)}{p-q} \\
&= \frac{px - qx - px + py + x - y}{p-q} \\
&= \frac{-qx + py + x - y}{p-q}
\end{aligned}
$$

$$
\begin{aligned}
t_n &= a + (n-1)d \\
&= \frac{-qx + py + x - y}{p-q} + (n-1)\left(\frac{x-y}{p-q}\right) \\
&= \frac{-qx + py + x - y + nx - ny - x + y}{p-q} \\
&= \frac{-qx + py + nx - ny}{p-q} \\
&= \frac{x(n-q) + y(p-n)}{p-q}
\end{aligned}
$$

Question 10

Let the first term be a and the common difference d.

Then $t_4 = a + 3d = 28$

$t_{16} = a + 15d \le 50$

$t_{17} = a + 16d > 50$.

From t_4, $d = \dfrac{28-a}{3}$.

Substitute into t_{16} and t_{17}.

Thus $a + 15\left(\dfrac{28-a}{3}\right) \le 50$ and $a + 16\left(\dfrac{28-a}{3}\right) > 50$

$a + 5(28 - a) \le 50$ and $3a + 16(28 - a) > 150$

$a + 140 - 5a \le 50$ and $3a + 448 - 16a > 150$

$90 \le 4a$ and $13a < 298$

$22\tfrac{1}{2} \le a < 22\tfrac{12}{13}$

Exercise 1.03 Sum of an arithmetic sequence

Question 1

a There are 10 terms.

First + last = 73

Sum = $\frac{10}{2}\times 73 = 365$

b $a = 82, d = -2$

$$t_n = a + (n-1)d$$
$$16 = 82 - 2(n-1)$$
$$n = 34$$

Sum = 17(82 + 16) = 1666

c $a = 8\frac{1}{2}, d = 1\frac{5}{6}$

$$t_n = a + (n-1)d$$
$$36 = \frac{17}{2} + (n-1)\times\frac{11}{6}$$
$$= \frac{40}{6} + \frac{11n}{6}$$

$$\frac{11n}{6} = \frac{88}{3}$$
$$n = 16$$

Sum = $8\left(8\frac{1}{2} + 36\right) = 356$

d $a = 1\frac{3}{8}, d = -\frac{1}{4}$

$$t_n = a + (n-1)d$$
$$-3\frac{3}{8} = \frac{11}{8} + (n-1)\times\left(-\frac{1}{4}\right)$$
$$-\frac{27}{8} = \frac{13}{8} - \frac{n}{4}$$
$$\frac{-40}{8} = -\frac{n}{4}$$
$$n = 20$$

Sum = $10\left(1\frac{3}{8} - 3\frac{3}{8}\right) = -20$

e $a = 9x - 3, d = 1 - 2x$

$$t_n = a + (n-1)d$$
$$5 - 7x = (9x - 3) + (n-1)(1 - 2x)$$
$$5 - 7x = 11x - 4 + n - 2xn$$
$$9 - 18x = n(1 - 2x)$$

$$n = \frac{9 - 18x}{1 - 2x}$$
$$= 9$$

Sum = $4.5(9x - 3 + 5 - 7x) = 9x + 9$

f $a = 16.3, d = 4.9$

$$t_n = a + (n-1)d$$
$$94.7 = 16.3 + 4.9(n-1)$$
$$83.3 = 4.9n$$

$$n = 17$$

Sum = 8.5(16.3 + 94.7) = 943.5

Question 2

a $a = -35, d = 3$

$$S_{30} = \frac{30}{2}\left[2\times(-35)+29\times 3\right] = 255$$

b $a = 21, d = -3\frac{4}{5}$

$$S_{40} = \frac{40}{2}\left[2\times 21 + 39\times\left(-3\frac{4}{5}\right)\right]$$
$$= -2124$$

c $a = 12x, d = 2 - x$

$$S_{20} = \frac{20}{2}\left[2\times 12x + 19(2-x)\right] = 50x + 380$$

d $a = 1.8, d = 1.3$

$$S_{60} = \frac{60}{2}\left[2\times 1.8 + 59\times 1.3\right] = 2409$$

e $a = -2.5, d = -2.8$

$$S_{40} = \frac{40}{2}\left[2\times(-2.5) + 39\times(-2.8)\right]$$
$$= -2284$$

f $a = \frac{14}{15}, d = 3\frac{1}{5}$

$$S_{50} = \frac{50}{2}\left[2\times\frac{14}{15} + 49\times 3\frac{1}{5}\right]$$
$$= 3966\frac{2}{3}$$

Question 3

$a = 9, d = 4.5$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$1000 = \frac{n}{2}\left[2\times 9 + (n-1)\times 4.8\right]$$
$$2000 = n\left[4.8n + 13.2\right]$$
$$4.8n^2 + 13.2n - 2000 = 0$$
$$n = \frac{-13.2 \pm \sqrt{(13.2)^2 - 4\times 4.8\times -2000}}{2\times 4.8}$$
$$= \frac{-13.2 \pm \sqrt{38574.24}}{9.6}$$
$$n \approx 19.1 \text{ or } -21.8$$

$$S_{19} = 9.5\left[2\times 9 + 18\times 4.5\right] = 940.5$$

$$S_{20} = 10\left[2\times 9 + 19\times 4.5\right] = 1055$$

20 terms are needed for the series to exceed 1000.

Question 4

$a = -45$, $d = 6$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$155 = \frac{n}{2}\left[2\times\left(-45\right) + (n-1)\times 6\right]$$
$$310 = n\left[6n - 96n\right]$$

$$6n^2 - 96n - 310 = 0$$
$$2\left(3n^2 - 48n - 155\right) = 0$$

$$n = \frac{48 \pm \sqrt{(-48)^2 - 4\times 3\times\left(-155\right)}}{2\times 3}$$
$$= \frac{48 \pm \sqrt{4164}}{6}$$
$$n \approx 18.8 \text{ or } -2.8$$

$$S_{18} = 9\left[2\times\left(-45\right) + 17\times 6\right] = 108$$

$$S_{19} = 9.5\left[2\times\left(-45\right) + 18\times 6\right] = 171$$

19 terms are needed for the series to exceed 155.

Question 5

Need to find the sum of all the positive terms in series.

$$a = 46, d = -6, t_n = 0$$

$$t_n = a + (n-1)d$$
$$0 = 46 + (n-1)\times\left(-6\right)$$

$$6n = 52$$
$$n = 8.66\ldots$$

$$t_8 = 46 + 7\times\left(-6\right) = 4$$
$$t_9 = 46 + 8\times\left(-6\right) = -2$$

$$\therefore S_8 = 4\left[2\times 46 + 7\times\left(-6\right)\right] = 200$$

Question 6

$$S_{20} = 46, a = 8$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$46 = 10\left[2\times 8 + 19d\right]$$

$$190d = -114$$
$$d = -\frac{2}{3}$$

$$S_{30} = 15\left[2\times 8 + 29\times\left(-\frac{3}{5}\right)\right] = -21$$

Question 7

$72 = a + 4d$

$-38 = a + 29d$

Solve simultaneously

$-110 = 25d$

$d = -4.4$

$72 = a + 4 \times -4.4$

$a = 89.6$

Require $S_n < 0$

$0 = \frac{n}{2}\left[2 \times 89.6 + (n-1) \times (-4.4)\right]$

$0 = n(183.6 - 4.4n)$

$n = 0$ or $n \approx 4.7$

$S_{41} = 20.5\left[2 \times 89.6 + 40 \times (-4.4)\right] = 65.6$

$S_{42} = 21\left[2 \times 89.6 + 41 \times (-4.4)\right] = -25.2$

$\therefore$ Require a minimum of 42 terms for sum to be negative.

Question 8

$S_{16} = 92,$ $S_{20} = 175$

$92 = 8\left[2a + 15 \times d\right]$ $175 = 10\left[2a + 19d\right]$

$92 = 16a + 120d$ $175 = 20a + 190d$

$35 = 4a + 38d$

Solve simultaneously:

$92 = 16a + 120d$

$140 = 16a + 152d$

$48 = 32d$

$\frac{48}{32} = d$

$d = 1\frac{1}{2}$

$35 = 4a + 38 \times 1\frac{1}{2}$

$4a = -22$

$a = -5\frac{1}{2}$

$S_{100} = 50\left[2 \times \left(-5\frac{1}{2}\right) + 99 \times 1\frac{1}{2}\right]$

$= 6875$

Question 9

$S_{40} = t_{40},\quad t_7 = 26$

$26 = a + 2d$

$S_{40} = 20\left[2a + 39d\right]$

$t_{40} = a + 39d$

$20\left[2a + 39d\right] = a + 39d$

$39a + 741d = 0$

Solve following equations simultaneously.

$(a + 6d = 26) \times 39$

$39a + 234d = 1014$

$39a + 741d = 0$

$507d = -1014$

$d = -2$

$a - 12 = 26$

$a = 38$

$S_{25} = 12.5\left[2 \times 38 + 24 \times (-2)\right]$

$= 350$

Question 10

$t_{10} = 18,\; S_{20} = 280$

$18 = a + 9d$

$280 = 10\left[2a + 19d\right]$

$28 = 2a + 19d$

Solve following equations simultaneously.

$(a + 9d = 18) \times 2$

$2a + 18d = 36$

$2a + 19d = 28$

$d = -8$

$a - 72 = 18$

$a = 90$

Let $S_n = 500$

$500 = \frac{n}{2}\left[2 \times 90 + (n-1) \times -8\right]$

$1000 = n\left[180 - 8n\right]$

$8n^2 - 180n + 1000 = 0$

$n = \frac{180 \pm \sqrt{(-180)^2 - 4 \times 8 \times 1000}}{2 \times 8} = \frac{180 \pm \sqrt{400}}{16}$

$n \approx 13.75 \text{ or } 8.75$

$S_8 = 4\left[2 \times 90 + 7 \times -8\right] = 496$

$S_9 = 4.5\left[2 \times 90 + 8 \times -8\right] = 522$

Check sum of terms until sum falls below 500 again.

$S_{10} = 540,\quad S_{11} = 550,\quad S_{12} = 552,\quad S_{13} = 546$

$S_{14} = 532,\quad S_{15} = 510,\quad S_{16} = 480$

$\therefore$ There are 7 terms of the series which exceed 500.

Exercise 1.04: Applications of arithmetic sequences and series

Question 1

$a = 20$, $d = -1$

$t_n = 8$

$8 = 20 + (n-1) \times -1$

$n = 13$

$S_{13} = 6.5\left[2 \times 20 + 12 \times (-1)\right] = 182$

Question 2

a Square numbers are: 1, 4, 9, 16, …

$\therefore$ Numbers added to give square numbers are: 1, 3, 5, 7, … (odd numbers)

b $t_1 = 1$,

$t_2 = 3 = 2 \times 2 - 1$

$t_3 = 5 = 2 \times 3 - 1$

$t_4 = 7 = 2 \times 4 - 1$

$\therefore\ t_n = 2n - 1$

$a = 1,\ l = 2n - 1$

$$
\begin{aligned}
S_n &= \frac{n}{2}(a + l) \\
&= \frac{n}{2}(1 + 2n - 1) \\
&= n^2
\end{aligned}
$$

Question 3

Pentagonal numbers are: 1, 5, 12, 22, 35, …

∴ Numbers added to give square numbers are: 1, 4, 7, 10, 13 …

$t_1 = 1$

$t_2 = 4 = 3\times 2 - 2$

$t_3 = 7 = 3\times 3 - 2$

$t_4 = 10 = 3\times 4 - 2$

$\therefore t_n = 3n - 2$

$a = 1,\ l = 3n - 2$

$$S_n = \frac{n}{2}(a+l)$$

$$= \frac{n}{2}(1 + 3n - 2)$$

$$= \frac{n}{2}(3n - 1)$$

$$= \frac{n(3n-1)}{2}$$

Question 4

Hexagonal numbers are: 1, 6, 15, 28, 45, …

∴ Numbers added to give square numbers are: 1, 5, 9, 13, 17, 21, … (odd numbers)

$t_1 = 1$

$t_2 = 5 = 4\times 1 + 1$

$t_3 = 9 = 4\times 2 + 1$

$t_4 = 13 = 4\times 3 + 1$

$t_5 = 17 = 4\times 4 + 1$

$\therefore t_n = 4(n-1) + 1$

$a = 1,\ l = 4(n-1) + 1$

$$S_n = \frac{n}{2}(a+l)$$

$$= \frac{n}{2}\left[1 + 4(n-1) + 1\right]$$

$$= \frac{n}{2}(4n - 2)$$

$$= 2n^2 - n$$

$$= n(2n - 1)$$

Question 5

a a = \$6000, d = 0.06 × \$6000 = \$360

(After 8 years, it will accumulate 7 years of interest.)

$$t_n = a + (n-1)d$$
$$t_8 = 6000 + 7 \times 360$$
$$= 8520$$

Grown to \$8520.

b Second \$6000 is added one year later.

$\therefore$ Savings for 7 years with 6 years’ interest.

$$t_8 = 6000 + 6 \times 360$$
$$= 8160$$

Grown to \$8160.

c a = 6000, d = 360

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$S_8 = 4\left[2 \times 6000 + 7 \times 360\right]$$
$$= 58\ 080$$

She has \$58 080 after 8 years.

Question 6

a 2500 – 1300 = 1200 L in tank after 1.5 hours.

$$\frac{1200}{1.5} = 800\,\text{L/h}$$

b 11.30 a.m. to 3 p.m. = 3.5 hours

800 × 3.5 = 2800 L

2500 + 2800 = 5300 L

c a = 1300, d = 800, $t_n = 7900$

$$t_n = a + (n-1)d$$

$$7900 = 1300 + (n-1)\times 800$$

$$7900 = 500 + 800n$$

$$800n = 7400$$

$$n = 9.25$$

Tank will be full after 9.25 hours = 9 hours 15 min
Time = 10 a.m. + 9 hours 15 min = 7:15 p.m.

Question 7

a = \$130 000, d = –\$8500

$$V = 130\,000 - 8500(n-1)$$

$$= 138\,500 - 8500n$$

$$t_n = a + (n-1)d$$

$$28\,000 = 138\,500 - 8500n$$

$$8500n = 110\,500$$

$$n = 13$$

∴ It will take 13 years for the value to fall to \$28 000.

Question 8

Sequence is 147, 147, 138, 138, 129, 129, 120, 120, …3

Find sum of sequence 147, 138, 129, 120, …, 3 and then double the answer.

a = 147, d = –9, $t_n = 3$

$t_n = a + (n - 10d$

$3 = 147 - 9(n - 1)$

$3 = 156 - 9n$

$9n = 153$

$n = 17$

$S_{17} = 8.5\left(147 + 3\right) = 1275$

Hence, the dog runs 2 × 1275 metres = 2550 metres or 2.55 km.

Question 1

a 1, 3, 7, 15, 31, 63, 127, … (adding 2, then 4, then 8, etc.)

$\therefore\ t_7 = 63 + 64 = 127$

b $t_8 = 127 + 128 = 255$

$t_9 = 255 + 256 = 511$

$\therefore\ t_{10} = 511 + 512 = 1023$

Question 2

a $t_1 = 1$

$t_2 = 2 \times 1 + 4 = 6$

$t_3 = 2 \times 6 + 4 = 16$

$t_4 = 2 \times 16 + 4 = 36$

$t_5 = 2 \times 36 + 4 = 76$

$\therefore$ Sequence is 1, 6, 16, 36, 76, …

b $t_1 = 1 = 5 \times 1 - 4 = 5 \times 2^0 - 4$

$t_2 = 6 = 5 \times 2 - 4 = 5 \times 2^1 - 4$

$t_3 = 16 = 5 \times 4 - 4 = 5 \times 2^2 - 4$

$t_4 = 36 = 5 \times 8 - 4 = 5 \times 2^3 - 4$

$t_5 = 76 = 5 \times 16 - 4 = 5 \times 2^4 - 4$

$\therefore\ t_n = 5 \times 2^{n-1} - 4$

Question 3

a 4 – 2 = 2, 8 – 4 =4, 16 –8 =8

$\therefore$ No common difference. Not an AP.

b 9 – 5 = 4, 13 – 9 = 4, 17 – 13 = 4

$\therefore$ An AP with common difference 4.

c 7 – 9 = –2, 4 – 7= –3, 0 – 4 = –4

$\therefore$ No common difference. Not an AP.

d 9 – 11 = –2, 7 – 9 = –2, 5 – 7 = –2

$\therefore$ An AP with common difference –2

Question 4

a $a = 14,\ d = -3\frac{2}{7}$

$t_n = a + (n-1)d$

$t_n = 14 + (n-1) \times -\frac{23}{7}$

$t_n = \frac{98}{7} - \frac{23n+23}{7}$

$t_n = \frac{121-23n}{7}$

b

$t_n = \frac{121-23n}{7}$

$t_{10} = \frac{121-23\times 10}{7}$

$= \frac{-109}{7}$

$t_n = -15\frac{4}{7}$

c $t_n = -30$

$t_n = a + (n-1)d$

$-30 = \frac{121-23n}{7}$

$-210 = 121 - 23n$

$23n = 331$

$n = 14.39$

$t_{14} = \frac{121-23\times 14}{7} = -28.7$

$t_{15} = \frac{121-23\times 15}{7} = -32$

$\therefore t_{15} = -32$ is the first term below –30.

Question 5

$t_9 = 28, \quad t_{15} = -6$

$a + 8d = 28$

$a + 14d = -6$

$6d = -34$

$d = \frac{-34}{6} = \frac{-17}{3}$

$a + 8 \times \left(-\frac{17}{3}\right) = 28$

$a = \frac{220}{3}$

Hence

$t_n = \frac{220}{3} - \frac{17(n-1)}{3}$

$= \frac{220 - 17(n-1)}{3}$

$t_3 = \frac{220 - 17 \times 2}{3}$

$= 62$

Question 6

$a = 21.9,\ d = 3.1,\ t_n = 65.3$

$t_n = a + (n-1)d$

$65.3 = 21.9 + 3.1(n-1)$

$= 18.8 + 3.1n$

$3.1n = 465$

$n = 15$

$S_n = \frac{n}{2}[a + l]$

$S_{15} = 7.5[21.9 + 65.3]$

$= 654$

Question 7

$a = 2p + 10q$

$d = (3p + 8q) - (2p + 10q)$

$= p - 2q$

$S_{30} = 15[2(2p + 10q) + 29(p - 2q)]$

$= 15[33p - 38q]$

$= 495p - 570q$

Question 8

$a = 21.5$, $d = 19.6 - 21.5 = -1.9$, $S_n > 130$

Let $S_n = 130$.

$$130 = \frac{n}{2}\left[2\times 21.5 - 1.9(n-1)\right]$$
$$260 = n\left[44.9 - 1.9\right]$$
$$260 = 44.9n - 1.9n^2$$

$$1.9n^2 - 44.9n + 260 = 0$$

$$n = \frac{44.9 \pm \sqrt{(-44.9)^2 - 4\times 1.9\times 260}}{2\times 1.9}$$
$$= \frac{44.9 \pm \sqrt{40.01}}{3.8}$$
$$n \approx 13.5 \text{ or } 10.1$$

Try values of n either side of 10.1 as asked for when series first exceeds 130.

$$S_{10} = 5\left[2\times 21.5 - 1.9\times 9\right] = 129.5$$
$$S_{11} = 4.5\left[2\times 21.5 - 1.9\times 10\right] = 132$$

$\therefore$ Require 11 terms.

Question 9

Sequence is: (4 × 35), (4 × 34), (4 × 33), (4 × 32), …
or 140, 136, 132, 128, …

$\therefore$ $a = 140$, $d = -4$

15 rows high, therefore t_{15} is the last term in the sequence.

$$t_n = a + nd$$
$$t_{15} = 140 - 14\times 4$$
$$= 84$$

$\therefore$ $S_{15} = 7.5\left[140 + 84\right] = 1680$

Question 10

Sequence is: 1, 4, 7, 10, 13, …

$\therefore\ a = 1, d = 3$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$= \frac{n}{2}\left[2 \times 1 + 3(n-1)\right]$$

$$= \frac{n}{2}\left[3n - 1\right]$$

$$= \frac{n(3n-1)}{2}$$

Question 11

$$t_8 = \frac{a}{2}$$

$$a + 7d = \frac{a}{2}$$

$$a - \frac{a}{2} = -7d$$

$$\frac{a}{2} = -7d$$

$$a = -14d$$

$$t_{22} = -20$$

$$a + 21d = -20$$

Substitute $a = -14d$ into the above equation.

$$-14d + 21d = -20$$

$$7d = -20$$

$$d = \frac{-20}{7}$$

$$a = -14 \times \left(-\frac{20}{7}\right)$$

$$= 40$$

$$\therefore t_{14} = 40 + 13 \times \left(-\frac{20}{7}\right) = \frac{20}{7} = 2\frac{6}{7}$$

Question 12

Sum of all numbers between 100 and 1000.

a = 100, *d* = 1, *l* = 1000

$$S_n = \frac{n}{2}\left[a + l\right]$$

$$S_{901} = 450.5\left[100 + 1000\right]$$

$$= 450.5 \times 1100$$

$$= 495\,550$$

Numbers between 100 and 1000 that are divisible by 11 are 110, 121, 132, 143, …

$\therefore$ a = 110, d = 11, $t_n < 1000$

$$t_n = a + (n-1)d$$

$$1000 = 110 + 11(n-1)$$

$$1000 = 99 + 11n$$

$$n = \frac{901}{11}$$

$$= 81.9$$

Use n = 81 (round down)

$$S_{81} = 40.5\left[2 \times 110 + 80 \times 11\right]$$

$$= 44\,550$$

$\therefore$ Sum of numbers not divisible by 11 is

495 550 – 44 550 = 451 000

Question 13

$$S_{21} = 10.5\left[2a + 20d\right]$$

$$S_{11} = 5.5\left[2a + 10d\right]$$

$$S_{21} = S_{11}$$

$$\therefore\ 10.5\left[2a + 20d\right] = 5.5\left[2a + 10d\right]$$

$$10a + 155d = 0 \qquad [1]$$

$$t_{63} = -12$$

$$a + 62d = -12 \qquad [2]$$

Solve equations [1] and [2] simultaneously.

[1] – 10 × [2] gives

$$-465d = 120$$

$$d = -\frac{8}{31}$$

$$a + 62 \times \left(-\frac{8}{31}\right) = -12$$

$$a = 4$$

$$S_{94} = 47\left[2 \times 4 + 93 \times \left(-\frac{8}{31}\right)\right]$$

$$= 47\left[8 - 24\right]$$

$$= -752$$

Question 14

Length of tree with branches = 20.6 – 2 = 18.6 m

Each branch is 0.6 m apart, so the number of branches = 18.6 ÷ 0.6 = 31

$a = 5,\ t_n = 0.2,\ n = 31,\ d = ?$

$0.2 = 5 + 30d$

$30d = -4.8$

$d = -0.16$

$S_{31} = 15.5\left[2 \times 5 + 30 \times (-16)\right]$

$= 80.6$

Number of leaves = 80.6 × 1500

= 120 900 leaves